

Modeling Multivariable Time Series Using Regular and Singular Autoregressions

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Rose petals let us scatter, and fill the cup with red wine.
The firmaments let us shatter, and come with a new design.

Hafez Shirazi (Persian poet, 1325-1389 AD)

To Sara

Declaration

My doctoral studies have been conducted under the guidance and supervision of Professor Brian D.O. Anderson, Professor Manfred Deistler and Doctor Jochen Trumppf. Most of the results in this thesis have been submitted to or published in refereed journals, scholarly book chapters, and presented at international conferences. Some of these results have been achieved in collaboration with other researchers. These results include:

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2. E. Felsenstein, B. Funovits, M. Deistler, B. D. O. Anderson, M. Zamani, W. Chen, Regular and singular multivariate autoregressive models and mixed frequency data: identifiability and consistent estimation, *Journal of Econometric Theory*, 2013 (submitted).
3. M. Zamani, B. D. O. Anderson, U. Helmke, and W. Chen, On the zeros of blocked time-invariant systems. *Systems and Control Letters*, 62(7), pages 597-603, 2013.
4. B. D. O. Anderson, M. Zamani, and G. Bottegal, On the zero properties of tall linear systems with single-rate and multirate outputs, *Mathematical System Theory-Festschrift in Honor of Uwe Helmke on the Occasion of his Sixtieth Birthday*, K. Hüper and J. Trumppf, Eds., pages 31-49, Create Space, 2013.
5. M. Zamani, U. Helmke, and B. D. O. Anderson, Zeros of networks of linear multi-agent systems: time-invariant interconnections, *Proceedings of the IEEE Conference on Decision and Control (CDC)*, pages 5939-5944, 2013.
6. M. Zamani, G. Bottegal, and B. D. O. Anderson, On the properties of linear multirate systems with coprime output rates, *Proceedings of the IEEE Conference on Decision and Control (CDC)*, pages 2734-2739, 2013.
7. M. Zamani, and B. D. O. Anderson, On modeling of tall linear systems with multirate outputs, *Proceedings of the Asian Control Conference (ASCC)*, 2013.
8. M. Zamani, E. Felsenstein, B. D. O. Anderson, and M. Deistler, Mixed frequency structured AR model identification, *Proceedings of the European Control Conference (ECC)*, pages 1928-1933, 2013.

9. B. D. O. Anderson, M. Deistler, E. Felsenstein, B. Funovits, P. Zadrozny, M. Eichler, W. Chen, and M. Zamani, Identifiability of regular and singular multivariate autoregressive models from mixed frequency data, Proceedings of the IEEE Conference on Decision and Control (CDC), pages 184-189, 2012.
10. M. Zamani, B. D. O. Anderson, U. Helmke, and W. Chen, On zeros of tall, fat and square blocked time-invariant systems, Proceedings of the International Symposium on Mathematical Theory of Networked and Systems (MTNS), 2012.
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12. M. Zamani, W. Chen, B. D. O. Anderson, M. Deistler and A. Filler, On the blocked systems of linear systems with mixed frequency data, Proceedings of the IEEE Conference on Decision and Control (CDC), pages 4312-4317, 2011.



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Mohsen Zamani,
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Abstract

The primary aim of this thesis is to study the modeling of high-dimensional time series with periodic missing observations. This study is very important in different branches of science and technology such as: econometric modeling, signal processing and systems and control. For instance, in the field of econometric modeling, it is crucial to provide proper models for national economies to help policy makers with decision making and policy adjustments. These models are built upon available high-dimensional data sets, which are not usually collected at the same rate. For example, some data such as, the employment rate are available on a monthly basis while some others like the gross domestic product (GDP) are collected quarterly. Motivated by applications in econometric modeling, we mainly consider systems, which have two sets of measurement streams, one stream being available at all times and the other one is observed every N -th time.

There are two major issues involved with modeling of high-dimensional time series with periodic missing observations, namely, the curse of dimensionality and missing observations. Generalized dynamic factor models (GDFMs), which have been recently introduced in the field of econometric modeling, are exploited to handle the curse of dimensionality phenomenon. Furthermore, the blocking technique from systems and control is used to tackle issues associated with the missing observations.

In this thesis, we consider a class of GDFMs and assume that there exists an underlying linear time-invariant system operating at the highest sample rate and our task is to identify this model from the available mixed frequency measurements. To this end, we first provide a very detailed study about zeros of linear systems with alternate missing measurements. Zeros of this class of linear systems are examined when the parameter matrices A, B, C and D in a minimal state space representation of a transfer function matrix $C(zI - A)^{-1}B + D$ corresponding to the underlying high frequency system assume generic values. Under this setting, we then illustrate situations under which linear systems with missing observations are completely zero-free. It is worthwhile noting that the obtained condition is very common in an econometric modeling context. Then we apply this result and assume that the underlying high frequency system has an autoregressive (AR) structure. Next, we study identifiability of AR systems from those population second order moments, which can be observed in principle. We propose the method of modified extended Yule-Walker equations to show that the set of identifiable AR systems is an open and dense subset of the associated parameter space i.e. AR systems are generically identifiable.

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Introduction

1.1 Motivation

In different branches of science and technology, one of the important problems is to analyze and model high-dimensional times series when some parts of observations are periodically missing. For instance, in the field of econometric modeling, analyzing and providing proper models for national economies allow policy adjustments by those with fiscal and monetary responsibilities. In this area, it is common to have some data which are collected monthly, while some other data may be obtained quarterly e.g. GDP data, or even annually ¹ Clements and Galvão [2008], Schumacher and Breitung [2006], in most advanced countries, the number of such time series generally exceeds 100.

In signal processing and systems and control, mixed frequency data arise naturally from multi-rate sampled data systems Meyer and Burrus [1975], Chen and Francis [1995].

This thesis is constructed on the intersection area of econometric modeling and systems and control and uses knowledge from both fields to **study problems involved with modeling high-dimensional time series with periodically missing observations**. In particular, the advanced technique of *generalized dynamic factor models* (GDFMs) from econometric modeling is used to handle the problems associated with the curse of dimensionality. Moreover, to tackle the problems associated with periodically missing observations, the well-developed technique of *blocking* or *lifting* is adopted from systems and control as described by Colaneri et al. [1990, 1992] and similarly for signal processing as described in Meyer and Burrus [1975] and Meyer [1990].

In the following pages, we first provide a review of the techniques used in econometric modeling for dealing with mixed frequency data. We then provide background information and review the literature associated with GDFMs. We then briefly explain the mathematical theory of a class of GDFMs studied in this thesis. Finally, we present the literature review regarding the blocking technique and multi-rate systems, which have been studied in systems and control and signal processing for quite a long time.

¹In econometric modeling context, the term mixed frequency data is used to refer to such a situation.

1.2 Mixed Frequency Data Analysis in Econometric Modeling

As mentioned earlier, one of the important challenges in econometric modeling is dealing with mixed frequency data. To handle this problem, several approaches like aggregation and interpolation, bridge equations, mixed data sampling (MIDAS), mixed autoregressive-moving-average (ARMA) models and mixed frequency factor models have been used in econometric modeling literature. In the following subsections, these approaches are briefly reviewed.

1.2.1 Aggregation and Interpolation

It is common in empirical applications to preprocess data such that all data are sampled at the same frequency. The prefiltering process involves either aggregation of the high frequency data to the lowest frequency, or interpolation of the low frequency data to the highest frequency. In most empirical applications, aggregation of high frequency data to the lowest frequency is done either by taking the average, sum or a representative value. It is important to note that in the aggregation process a large portion of useful information can be destroyed; thus, the aggregation approach is not quite satisfactory Lütkepohl [1987]. The interpolation of low frequency data has been done in different ways, see Angelini et al. [2006] and Lanning [1986], but, the common way is to first interpolate the missing data and then use this new information to estimate the model parameters using techniques like the Kalman filter.

1.2.2 Bridge Equations

Another approach which attempts to combine mixed frequency time series is the so called bridge equations (see Diron [2008] and Baffigi et al. [2004]). Under the situation where there exists two streams of data, namely low frequency data and high frequency data, the bridge equations are linear regressions that establish a bridge between high frequency data and those low frequency ones. The bridge models are essentially established at the low frequency. In this approach, the high frequency data are predicted up to a certain predication horizon from a separate time series model. Then the obtained predictions are processed such that they can be used in a low frequency model. For more extensive studies on bridge models interested readers can refer to Foroni and Marcellino [2013] and the references listed therein.

1.2.3 MIDAS

This type of model was initially proposed by Ghysels et al. [2004] but later extensions of that have been introduced in the literature (check Clements and Galvão [2005]). In the MIDAS model, the low frequency dependent variable is expressed as the sum of distributed lag of high frequency explanatory variable and an error term. Later, Clements and Galvão [2005] modified this model by introducing an autoregressive

term. Since this type of model is not our focus in this thesis, further details are not provided and we refer readers to Wohlrabe [2009] for a more extensive study about MIDAS models.

1.2.4 Mixed Frequency ARMA

Another interesting approach for handling mixed frequency data in the continuous time framework was proposed initially in Zadrozny [1988]. This was later extended to the discrete time domain in Zadrozny [1990]. In Zadrozny [1990], the author assumed that there exists a high frequency underlying model which gives rise to all data; he then exploited the state space and Kalman filtering approach to deal with mixed frequency ARMA model. Later, the authors of Chen and Zadrozny [1998] proposed a technique called extended Yule-Walker equations to estimate parameters of an autoregressive (AR) model with a mixed frequency observation vector. In Chen and Zadrozny [1998], the authors used available covariance data to construct Yule-Walker equations. The parameters were then estimated using the least square method.

1.2.5 Mixed Frequency Factor Models

Factor models have also been used in the literature for handling mixed frequency data. For instance, in Mariano and Murasawa [2003], the authors exploited a dynamic factor model for accommodating mixed frequency time series (monthly and quarterly data sets). There, a modification to the maximum likelihood method was proposed for estimation of the parameters. Further research in this direction is recorded in Giannone et al. [2008], where the authors suggested a two-step procedure involving the principal component calculation at the first stage and a Kalman smoother at the second stage. First, the factors are estimated using a data set with no missing entries i.e. high frequency data, then the Kalman smoother is applied to update the estimate using the mixed frequency data.

1.3 Generalized Dynamic Factor Models - Background

Analysis of high-dimensional time series is a highly important subject in many disciplines, for example: in time series analysis, econometric modeling as well as biomedical engineering. In this setup, not only are the number of collected samples large but also, the cross-sectional dimension is huge as well. It is known that the classical models such as AR or ARMA are not suitable for analyzing time series when the cross-sectional dimension is very large Pasching [2010]. This is due to the fact that dimensions of the parameter space for those classes of models are proportional to the square of the cross-sectional dimension and so it becomes obvious that the estimation of parameters is not practical. The term *curse of dimensionality* is used to refer to this problem. A common possible solution to this problem is to reduce the cross-sectional dimension by selecting a subset of measured variables; however, not

only will some of the information be lost but also this may lead to problem of overfitting White [2000]. Factor models, on the other hand, are known to be able to handle data sets with large cross-sectional dimension, as their parameter space dimension is linearly related to cross-sectional dimension.

Factor models were introduced by psychologists in the early twentieth century. Spearman [1904] and Burt [1909] exploited factor models in their mental ability tests and introduced a common factor, called *general intelligence*, which they claimed would drive the results in such a test. The results were then generalized in Thurstone [1931], where the author allowed for more than one factor.

The idea of factor models was then further generalized for modeling of multivariate time series by Geweke [1977], Sargent and Sims [1977] and Brillinger [1981].

In particular, a factor model can be generally characterized as

$$z_t = y_t + e_t, \quad t \in \mathbb{Z}, \quad (1.1)$$

where z_t is the observed process vector, e_t is the measurement noise and vector y_t is called *latent variable*. The main idea is that the latent variable i.e. y_t , can be expressed by a lower dimension common factor, say u_t . Moreover, e_t is not correlated with either u_t or y_t .

In the early works of Spearman [1904] and Burt [1909], the measurements are assumed to be independent and identically distributed. In addition, the measurement noise has a diagonal covariance matrix. Later, in works of Geweke [1977], Sargent and Sims [1977] and Scherrer and Deistler [1998], dynamic models were used to relate the latent variables to the driving factors and with measurement noise having a diagonal covariance matrix (*dynamic factor model*). Generalizing in a different direction, Chamberlain and Rothschild [1983] considered models with static latent factors i.e. y_t and u_t as being related in a static manner; they let the measurement noise be correlated in the cross-section (*generalized factor model*).

Almost 90 years after the introduction of the factor models, the theory of GDFMs was developed by Forni et al. [2000, 2005] and Stock and Watson [2002a,b]. GDFMs are dynamic factor models which allow weak correlation among the components of the measurement noise Zinner [2008], Filler [2010]. GDFMs have been mainly used in econometric modeling and finance applications but these models have great potential to be used in engineering as well; for instance in *oversensing* where a network of sensors collaboratively collect physical or environmental data Roemer and Mattern [2004]. Moreover, very recently there is evidence of heightened interest in systems and control toward GDFMs, see Bottegal and Picci [2011], Bottegal [2013] and the references listed therein.

Since a preliminary knowledge about GDFMs is needed in the upcoming chapters, in the next section, we briefly review the mathematical theory of a class of GDFMs used in this thesis.

1.4 Generalized Dynamic Factor Models - A Short Review

In this section, we introduce some of the preliminary assumptions and definitions for the class of GDFMs which are required in this thesis. More extensive details about GDFMs theory can be found in Filler [2010], Forni et al. [2000], Stock and Watson [2002a], Pasching [2010] and the references therein.

Let \mathcal{N} denote the measurement process dimension in (1.1); as stated earlier in (1.1), in factor model analysis, the measurement process z_t can be expressed as the sum of two processes, which are not observed i.e. e_t and y_t . We now state several core assumptions in defining GDFMs. It is also an essential assumption (see Stock and Watson [2002a] and Forni et al. [2000]) that both \mathcal{N} and $T \rightarrow \infty$, where T is the sample size. We now state the assumptions imposed on (1.1).

Assumption 1.4.1. 1. Both y_t and e_t are wide sense stationary with absolutely summable covariance.

$$2. \mathbb{E}[e_t] = \mathbb{E}[y_t] = 0 \quad \forall t.$$

$$3. \mathbb{E}[y_t e_t^T] = 0 \quad \forall t, s.$$

Using the above assumption, one can assume that spectral densities exist; and, in an obvious notation one can easily write

$$f_z(\lambda) = f_y(\lambda) + f_e(\lambda). \quad (1.2)$$

Assumption 1.4.2. There is an \mathcal{N}_0 such that for all $\mathcal{N}_0 \leq \mathcal{N}$, the spectral density $f_y(\lambda)$ is rational and has rank m , for all $\lambda \in [-\pi, \pi]$.

Assumption 1.4.3. The dimension of minimal state space realization, say n , of stable and minimum-phase spectral factor of $f_y(\lambda)$ is independent of \mathcal{N} ($\mathcal{N} \geq \text{some } \mathcal{N}_0$).

Assumption 1.4.4. The largest eigenvalue of $f_e(\lambda)$ is uniformly bounded in λ and \mathcal{N} .

Assumption 1.4.5. The m largest eigenvalues of $f_y(\lambda)$ diverge to infinity, $\forall \lambda$, as $\mathcal{N} \rightarrow \infty$.

The assumption 1.4.4 implies that the influence of measurement noise can be neglected by letting $\mathcal{N} \rightarrow \infty$. It is worthwhile noting that letting $\mathcal{N} \rightarrow \infty$ is a reasonable assumption as for many applications in which we are interested, \mathcal{N} is large Amengual and Watson [2007], Stock and Watson [2002a].

At this stage, we need to recall the following theorem from Hannan and Deistler [2012] and Deistler et al. [2010a], which explains the spectral factorization of $f_y(\lambda)$

Theorem 1.4.6. Every $\mathcal{N} \times \mathcal{N}$ rational spectral density $f_y(\lambda)$ of constant rank m for all $\lambda \in [-\pi, \pi]$ can be factorized as

$$f_y(\lambda) = (2\pi)^{-1} W(e^{-i\lambda}) W(e^{-i\lambda})^* \quad (1.3)$$

where $W(q)$, $q \in \mathbb{C}$ an also used as backward shift operator, is $\mathcal{N} \times m$ called stable minimum-phase spectral factor and has full-column rank with no poles and zeros for $|q| \leq 1$. Moreover,

* symbolizes the complex conjugate transpose. Furthermore, the stable minimum-phase spectral factor $W(q)$ is unique up to right multiplication by a constant orthogonal matrix of the proper size.

Now, one can use the Wold decomposition Rozanov [1967] and write y_t as

$$y_t = W(q)u_t = \sum_{j=0}^{\infty} W_j u_{t-j}, \quad \sum_{j=0}^{\infty} \|W_j\| < \infty, \quad (1.4)$$

where u_t is white noise signal of size m , $m < \mathcal{N}$, and $\mathbb{E}[u_t u_t^T] = 2\pi I_m$. In (1.4) the latent factors i.e. y_t , is obtained through a dynamic linear transformation from u_t . So, in an econometric context, u_t is called a *dynamic factor*. Moreover, no dynamic factor exist with dimension less than that of u_t , such that it expresses y_t by a linear dynamic systems. Hence, u_t is sometimes refereed to a *minimal dynamic factor* as well.

One of the important problems involved with GDFMs is the identification of the underlying model from data which is a complicated task in general Deistler and Hamann [2005], Forni et al. [2000] and Giannone et al. [2008]. After introducing some essential results in the early chapters of this thesis, we study this problem in great detail in Chapter 6. In the single rate setting i.e. monthly data only, the authors Stock and Watson [2002a,b] used the principal component analysis to achieve this task. Later, the authors in Deistler et al. [2010b], Filler [2010] and Anderson and Deistler [2008] showed that the set of *zero-free* transfer function matrices is an open and dense subset of the parameter space, which means that the transfer function matrix $W(q)$ is generically zero-free. This has the key consequence that identification of the model data (assuming a white noise input) becomes very simpler than for a normal case, as the system parameters can be identified through linear calculations from the observed data, by using a set of equations known as the *Yule-Walker* equations Lütkepohl [2005]. However, similar demonstration has been lacking so far in the mixed frequency case and one of the central contributions of the thesis is to address this shortcoming. In particular, we use the blocking technique to tackle this problem. The next section reviews some of the results in this area.

1.5 Blocking - Lifting in Signal Processing and Systems and Control

The well-known technique of blocking or lifting was developed in systems and control Chen and Francis [1995] and signal processing Vaidyanathan [1993]. In the systems and control literature, this method has mostly been exploited to transform linear discrete-time periodic systems to linear time-invariant systems so that the well-developed tools for linear time-invariant systems can be extended for design and analysis of linear discrete-time periodic systems Bolzern et al. [1986], Grasselli and Longhi [1988], Bittanti [1986] and Bittanti and Colaneri [2009]. For example, the authors in Bolzern et al. [1986] and Grasselli and Longhi [1988] extended the notions of poles and zeros of linear time-invariant systems to linear periodic systems. Some

necessary and sufficient conditions for structural properties such as observability and reachability were studied in Grasselli and Longhi [1991a] and Colaneri and Longhi [1995]. Moreover, the realization problem is recorded in Colaneri and Longhi [1995] and the related references recorded therein.

The blocking technique was applied to linear time-invariant systems as well, see Chen and Francis [1995], Khargonekar et al. [1985] and the references therein. For instance, in Khargonekar et al. [1985], linear time-invariant systems were blocked for the purpose of designing periodic controllers while the authors Chen and Francis [1995], performed the blocking technique on linear time-invariant systems for the purpose of dealing with multirate sampled-data systems.

Poles blocked systems were explored in Bittanti and Colaneri [2009] and Khargonekar et al. [1985]. Moreover, the authors Bittanti and Colaneri [2009] studied zeros of blocked systems obtained from the blocking of linear periodic systems. The results show that the blocked system has a finite zero if the related linear time-invariant unblocked system has a finite zero, which is a form of sufficiency result. However, this reference does not provide a necessary condition for the blocked system to have a finite zero; also, zeros at infinity are not considered. These gaps were covered in our works Chen et al. [2012] and Zamani et al. [2011], where we introduced some additional results about zeros of blocked systems. For instance, in Chen et al. [2012], matrix fraction descriptions (MFDs) were adopted to establish a relation between zeros of blocked systems and those of their corresponding unblocked systems. Moreover, in Zamani et al. [2011], the time domain approach was exploited to explore the zero properties of blocked systems. Both Chen et al. [2012] and Zamani et al. [2011] only considered *tall* blocked systems i.e. blocked systems with more outputs than inputs, and they showed that tall blocked systems have a zero if and only if the corresponding unblocked systems have a zero. Furthermore, in Chen et al. [2012] and Zamani et al. [2011] only blocked systems for which their associated transfer functions have *full-column normal rank*, were examined. Later, in Zamani et al. [2013a] the authors obtained more general results by relaxing the assumptions made in Zamani et al. [2011] and Chen et al. [2012] on the normal rank and the structure of the transfer function matrices. While the references Zamani et al. [2011, 2013a] and Chen et al. [2012] mainly considered unblocked linear time-invariant systems, as opposed to multirate systems; in Zamani and Anderson [2012], zeros of a class of unblocked multirate linear systems were explored. It was shown that the tall blocked systems obtained from the blocking of multirate systems with generic parameter matrices have *no finite nonzero zeros*.

1.6 Thesis Outline and Contributions

This thesis consists of seven chapters. The technical chapters can be arranged into two segments. The first segment consists of chapters two, three and four, and the second segment includes chapters five and six. The first segment of the current thesis is concerned with zeros of blocked systems and shows that these systems are

generically zero-free except possibly for the origin or infinity. This has the immediate consequence that the associated transfer function matrix can be generically expressed as an AR model. Hence, the second segment of this thesis focuses on AR models. In the rest of this section, the organization of thesis is set out in more detail and the contributions of each chapter are briefly summarized.

Chapter 2: On the Zeros of Blocked Time-invariant Systems

This chapter studies zeros of blocked linear systems resulting from the blocking of linear time-invariant systems. The main idea is to establish a relationship between zeros of blocked systems and zeros of their corresponding unblocked systems. In particular, it is shown that the blocked system has a zero if and only if the associated unblocked system has a zero. Furthermore, zeros of blocked systems are examined for a generic choice of matrices A, B, C and D in a minimal state space representation corresponding to the nonzero, nonconstant transfer function $D + C(zI - A)^{-1}B$. It is demonstrated that nonsquare blocked systems i.e. blocked systems with a number of outputs unequal to the number of inputs, generically have no zeros; however, square blocked systems i.e. blocked systems with equal number of inputs and outputs, generically have only finite zeros and these finite zeros have geometric multiplicity one. The results of this chapter are based on the published papers Zamani et al. [2013a] and Zamani et al. [2012].

Chapter 3: On the Normal rank and Zero-freeness of Tall Multirate Systems with Fast Outputs at the Fundamental Rate

In this chapter, tall discrete-time linear systems with multirate outputs are studied. In particular, the case where two output streams exist; one available at all times and the other one available every N times, is investigated and the attention is on zeros of this type of linear systems. In the systems and control literature, zeros of multirate systems are defined as those of their corresponding time-invariant blocked systems. Hence, zeros of tall blocked systems resulting from the blocking of linear systems with multirate outputs are mainly explored in this chapter. We specifically investigate zeros of tall blocked systems formed by blocking tall multirate linear systems with generic parameter matrices appearing in a minimal state space description. It is demonstrated that tall blocked systems generically have no finite nonzero zeros; however, they may have zeros at the origin or at infinity depending on the choice of the blocking delay and the input, state and output dimensions. The results of this chapter appear in Zamani and Anderson [2012] and Zamani et al. .

Chapter 4: On the Zero-freeness of Tall Multirate Systems with Coprime Output Rates

This chapter explores discrete-time linear systems with multirate outputs, assuming that two measured output streams are available at coprime rates. In the literature

these types of systems, which can be considered as periodic time-varying, are commonly studied in their blocked versions, since the well-known techniques of analysis developed for linear time-invariant systems can be used. In particular, we focus on some structural properties of the blocked systems and we prove that, under a generic setting i.e. for a generic choice of parameter matrices, the blocked systems are minimal. Moreover, we focus on zeros of tall blocked systems i.e. blocked systems with more outputs than inputs. In particular, we study those cases where the associated system matrix attains full-column rank. We examine situations where they generically have no finite nonzero zeros. The results of this chapter are presented in Zamani et al. [2013b].

Chapter 5: On the Identifiability of Regular and Singular Autoregressive Models from Mixed Frequency Data

This chapter is concerned with identifiability of an underlying high frequency multivariate AR system from mixed frequency observations. Such problems arise for instance in economics when some variables are observed monthly, whereas others are observed quarterly. If we have identifiability, the system and noise parameters and thus all second moments of the output process can be estimated consistently from mixed frequency data. Then, linear least squares methods for forecasting and interpolating nonobserved output variables can be applied. The focus is on general AR model and generic identifiability of such a model are shown using a modified extended Yule-Walker approach. The results of this chapter are based on the papers Anderson et al. [2012] and Felsenstein et al..

Chapter 6: On the Identifiability of Singular Autoregressive Models from Mixed Frequency Data-Linearly Dependent Lags

This chapter investigates the identifiability of an underlying high frequency multivariate stable singular AR system from mixed frequency observations. In particular, this chapter studies stable singular AR systems, where the covariance matrix associated with the vector obtained by stacking observation vector, y_t , and its lags from the first lag to the p -th one (p is the order of the AR system), is also singular. To deal with this, it is assumed that the column degrees of the associated polynomial matrix are known. We first consider that there are given nonzero unequal column degrees and we show generic identifiability of the system and noise parameters. Then we extend the results to allow zero column degrees in the polynomial matrix. In this case, we first show generic identifiability of the subsystem of the components with a nonzero column degree. Then we demonstrate how to obtain those components of the parameter matrices of the components corresponding to zero column degree. The results of this chapter are based on the published paper Zamani et al. [2013c].

Chapter 7: Conclusions and Future Research

This chapter provides concluding remarks and reviews contributions of this thesis and comments on some possible research directions for future study.

On the Zeros of Blocked Time-invariant Systems

Abstract

In this chapter, zeros of blocked linear systems resulting from the blocking of linear time-invariant systems are explored. Here, the main idea is to relate zeros of blocked systems to those of the corresponding unblocked systems. It is clearly demonstrated that the blocked system has a zero if and only if the associated unblocked system has a zero. Furthermore, zeros of blocked systems under a generic setting i.e. a setting in which parameter matrices A, B, C and D assume generic values, are examined. It is demonstrated that nonsquare blocked systems i.e. blocked systems with the number of outputs unequal to the number of inputs, generically have no zeros; however, square blocked systems i.e. blocked systems with equal number of inputs and outputs, generically have only finite zeros. It is worthwhile mentioning that even though this chapter focuses on unblocked linear time-invariant systems but, its results are very useful for what comes in the next chapter, where linear systems with alternately missing measurements are discussed.

2.1 Introduction

The well-known technique of blocking or lifting is an important tool in both systems and control Bittanti and Colaneri [2009] and signal processing Mitra [2000]. In this chapter, zeros of blocked systems resulting from the blocking of linear time-invariant systems are examined. It is worthwhile mentioning that zeros of unblocked linear time-invariant systems have been studied for a long time in the systems and control literature. Early work in this direction is due to Rosenbrock [1974], who emphasized the relevance of multivariable zeros in systems theory. Further efforts in this direction can be found in Wonham [1979], Karcaniyas and Kouvaritakis [1979], Kailath [1980], Karcaniyas and Vafiadis [2002], Filler [2010], Christou et al. [2010] and Mitrouli and Karcaniyas [1993]. For instance, Wonham [1979] studied zeros from a geometric perspective and showed that the existence of a nontrivial great common divisor among a set of polynomials is a nongeneric property. Furthermore, zeros of unblocked sys-

tems were also examined under a genericity assumption in Kailath [1980] and Filler [2010]. The authors showed that nonsquare unblocked systems i.e. unblocked systems with the number of outputs unequal to the number of inputs, are generically zero-free Filler [2010], Karcianas and Kouvaritakis [1979] and Kailath [1980]. Filler [2010], exploited the state space approach to show that tall unblocked systems i.e. unblocked systems with more numbers of outputs than inputs, when the defining matrices A, B, C and D in the minimal state space representation of transfer function matrix $C(zI - A^{-1})B + D$ assume generic values, are zero-free; Karcianas and Kouvaritakis [1979] used the matrix pencil to state the same result. However, Kailath [1980] recorded this result without any proof (Kailath [1980] page 448). Moreover, square unblocked systems i.e. unblocked systems with an equal number of inputs and outputs, generically have only finite zeros Karcianas and Vafiadis [2002] and Filler [2010].

Even though there are several papers dealing with zeros of unblocked linear time-invariant systems, to the author's best knowledge, the zeros of blocked time-invariant systems have been studied in few works (these works were reviewed in Subsection 1.5). It is worthwhile mentioning here that the analysis of zeros for blocked time-invariant systems is more complicated than that of unblocked systems. Firstly, when dealing with zeros of blocked systems, one has to be careful to distinguish different types of zeros i.e. zeros at the origin, finite nonzero zeros and zeros at infinity. The reason is that there may exist some delay caused by blocking, so occurrence of zeros at infinity or at the origin may happen. Moreover, as shown in Bittanti and Colaneri [2009], finite nonzero zeros of different blocked systems obtained from blocking of the same underlying unblocked multirate system can be related but this does not hold for zeros at the origin or zeros at infinity. Furthermore, as will be shown later, different type of zeros require special treatments. Secondly, the parameter matrices of blocked systems are structured and their entries cannot then be independently assigned. Thus, the results reviewed in the previous paragraph cannot be immediately applied to the blocked case.

In this chapter, we generalize the results of Chen et al. [2012] and Zamani et al. [2011]. Here, there exists no assumption such as tallness or fatness on the structure of blocked systems. Furthermore, we relax the assumption used in Chen et al. [2012] and Zamani et al. [2011] on the normal rank of the transfer function associated with the blocked system. In both aforementioned references, the normal rank of the transfer function is assumed to be equal to its number of columns; however, in the current chapter, no assumption on the normal rank of the transfer function associated with the blocked system is made. As mentioned earlier, due to the complicated nature of the problem under study, different treatments are required to tackle different types of zeros. Thus, finite nonzero system zeros, system zeros at zero and system zeros at infinity are studied separately. It is clearly shown that for all possible choices of zeros, the blocked system has a zero if and only if its corresponding unblocked system has a zero.

It is particularly shown here that nonsquare blocked systems i.e. blocked systems with number of outputs unequal to the number of inputs, when parameter matrices

A, B, C and D assume generic values have no zeros. However, when one is considering generic square blocked systems i.e. blocked systems with an equal number of inputs and outputs and the matrices A, B, C and D take generic values; it is illustrated that square blocked systems have no infinite zeros but they do have finite zeros, and the kernel of the system matrix associated with any zero is one dimensional.

The structure of this chapter is as follows. First, in Section 2.2, the formulation of the problem under study is given. Then a relationship between the transfer function of blocked systems and the transfer function of the associated unblocked systems is recalled. Based on the relationship obtained in Section 2.2, Section 2.3 relates zeros of blocked systems to those of their corresponding unblocked systems. Section 2.3 also studies zeros of blocked systems under the genericity assumption. Finally, Section 2.4 provides the concluding remarks.

2.2 Blocked Systems and Unblocked Systems - the State Space Representation and the Transfer Function

In this section, first a formulation for the problem under study is given. Then a relationship between the unblocked system transfer function and the blocked system transfer function is established. The obtained relationship is then used in the next section for the analysis of the blocked system zeros.

2.2.1 Unblocked Systems and Blocked Systems - the State Space

The linear time-invariant unblocked system under consideration is described as

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, \\ y_t &= Cx_t + Du_t, \end{aligned} \quad (2.1)$$

where $t \in \mathbb{Z}$, $x_t \in \mathbb{R}^n$, $y_t \in \mathbb{R}^p$ and $u_t \in \mathbb{R}^m$. Also, the transfer function associated with system (2.1) is defined as

$$W(z) = D + C(zI - A)^{-1}B, \quad (2.2)$$

where z is a forward shift operator i.e. $zu_t = u_{t+1}$, $zx_t = x_{t+1}$ and $zy_t = y_{t+1}$, and also represents a complex number.

Now we define for a fixed but arbitrary positive number $N > 1$

$$U_t = \begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+N-1} \end{bmatrix}, \quad Y_t = \begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+N-1} \end{bmatrix}, \quad (2.3)$$

where $t = 0, N, 2N, \dots$

Then the blocked system is given by Bolzern et al. [1986]

$$\begin{aligned} x_{t+N} &= A_b x_t + B_b U_t, \\ Y_t &= C_b x_t + D_b U_t, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} A_b &= A^N, \quad B_b = [A^{N-1}B \quad A^{N-2}B \quad \dots \quad B], \\ C_b &= [C^T \quad A^T C^T \quad \dots \quad A^{(N-1)T} C^T]^T, \\ D_b &= \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \dots & D \end{bmatrix}. \end{aligned} \quad (2.5)$$

An operator Z is defined such that $Zx_t = x_{t+N}$, $ZU_t = U_{t+N}$, $ZY_t = Y_{t+N}$. The symbol Z is also used to denote a complex value. Then the transfer function of (2.4) is denoted by

$$W_b(Z) = D_b + C_b(ZI - A_b)^{-1}B_b. \quad (2.6)$$

Furthermore, it is worthwhile remarking that the unblocked system (2.1) is a minimal realization of $W(z)$ if and only if the blocked system (2.4) is a minimal realization of $W_b(Z)$ Bittanti and Colaneri [2009].

2.2.2 Unblocked Systems and Blocked Systems - the Transfer Function

In the previous subsection, the state space representation for both unblocked and blocked systems was recalled Khargonekar et al. [1985], Burrus [1972] and Chen et al. [2012]. The aim of this subsection is to recall a relationship between $W_b(Z)$ and $W(z)$. The well-known result of Bittanti and Colaneri [2009] and Khargonekar et al. [1985] is summarized as the theorem below.

Theorem 2.2.1. *Consider the unblocked system (2.1) with transfer function $W(z)$ and the blocked system (2.4) with transfer function $W_b(Z)$. Then*

$$W_b(Z) = \begin{bmatrix} V_1(Z) & Z^{-1}V_N(Z) & Z^{-1}V_{N-1}(Z) & \dots & Z^{-1}V_2(Z) \\ V_2(Z) & V_1(Z) & Z^{-1}V_N(Z) & \dots & Z^{-1}V_3(Z) \\ V_3(Z) & V_2(Z) & V_1(Z) & \dots & Z^{-1}V_4(Z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ V_N(Z) & V_{N-1}(Z) & V_{N-2}(Z) & \dots & V_1(Z) \end{bmatrix} \quad (2.7)$$

and

$$W(z) = V_1(z^N) + z^{-1}V_2(z^N) + \dots + z^{-(N-1)}V_N(z^N), \quad (2.8)$$

where $V_1(Z) = D + C(ZI - A^N)^{-1}A^{N-1}B$ and $V_l(Z) = CA^{l-2}B + C(ZI - A^N)^{-1}A^{N+l-2}B$, $l = 2, \dots, N$.

Another important result regarding the relationship between $W_b(Z)$ and $W(z)$ is recorded in Burrus [1972] and Chen et al. [2012]. Assume that the transfer function of the unblocked system (2.1) is represented by a polynomial left coprime matrix fraction description (MFD) as

$$W(z) = Q^{-1}(z)P(z), \quad (2.9)$$

where

$$\begin{aligned} P(z) &= P_\mu + P_{\mu-1}z + \cdots + P_0z^\mu, \\ Q(z) &= Q_\mu + Q_{\mu-1}z + \cdots + Q_0z^\mu. \end{aligned} \quad (2.10)$$

In the above equation, μ is defined such that P_0 and Q_0 are not both zero. By coprimeness, P_μ and Q_μ are not both zero. Then it can be easily shown that associated with the blocked system there exists a transfer function with a polynomial left matrix fraction description as below

$$Y_k = W_b(Z)U_k, \quad W_b(Z) = \mathcal{A}^{-1}(Z)\mathcal{B}(Z), \quad (2.11)$$

where

$$\begin{aligned} \mathcal{A}(Z) &= \mathcal{A}_0 + \mathcal{A}_1Z + \cdots + \mathcal{A}_\alpha Z^\alpha + \mathcal{A}_{\alpha+1}Z^{\alpha+1}, \\ \mathcal{B}(Z) &= \mathcal{B}_0 + \mathcal{B}_1Z + \cdots + \mathcal{B}_\alpha Z^\alpha + \mathcal{B}_{\alpha+1}Z^{\alpha+1}, \end{aligned} \quad (2.12)$$

where α is the greatest integer less than μ/N and $\mathcal{A}_i, \mathcal{B}_i, i \in \{0, 1, \dots, \alpha + 1\}$ are constant coefficient matrices of size $N(p \times m)$ obtained by a certain procedure from the coefficient matrices $P_i, Q_i, i \in \{0, 1, \dots, \mu\}$, respectively Chen et al. [2012].

In the above, $W_b(Z)$ and $W(z)$ were related. However, by using the above calculation linking the \mathcal{B}_i to P_i , $\mathcal{B}(Z)$ and $P(z)$ can be as well associated. The following lemma is adapted from Lemma 2 in Chen et al. [2012] and relates the $\mathcal{B}(Z)$ and $P(z)$.

Lemma 2.2.2. *For a nonzero complex number Z_0 , let $z_i, i = 1, 2, \dots, N$, be N distinct complex numbers such that $z_i^N = Z_0, i = 1, 2, \dots, N$*

$$\begin{aligned} Y &= \begin{bmatrix} I_m & I_m & \cdots & I_m \\ z_1 I_m & z_2 I_m & \cdots & z_N I_m \\ \vdots & \ddots & \ddots & \vdots \\ z_1^{N-1} I_m & z_2^{N-1} I_m & \cdots & z_N^{N-1} I_m \end{bmatrix}, \\ \Lambda &= \begin{bmatrix} P(z_1) & P(z_2) & \cdots & P(z_N) \\ z_1 P(z_1) & z_2 P(z_2) & \cdots & z_N P(z_N) \\ \vdots & \ddots & \ddots & \vdots \\ z_1^{N-1} P(z_1) & z_2^{N-1} P(z_2) & \cdots & z_N^{N-1} P(z_N) \end{bmatrix}. \end{aligned} \quad (2.13)$$

Then

$$\mathcal{B}(Z_0)Y = \Lambda. \quad (2.14)$$

Proof. The proof can be done in a similar way as the proof of Lemma 2 in Chen et al. [2012]. \square

Note that the matrix Y has a special structure as it is a Kronecker product of a Vandermonde matrix with the identity matrix.

The results obtained in this section are helpful for analyzing zeros of the blocked system (2.4) in the following section.

2.3 Zeros of Blocked Systems

In this section, the definitions for zeros of the systems (2.1) and (2.4) are first reviewed. Then, zeros of blocked systems are studied. Since the analysis of zeros for blocked systems is quite complicated, three separate cases are considered, that is, 1) finite nonzero system zeros; 2) system zeros at infinity; and 3) system zeros at zero. Finally, the last subsection covers results for zeros of the system (2.4) with a generic choice of matrices A, B , etc.

2.3.1 Definition

In order to study zeros of the system (2.4), in the following, a precise definition for zeros of the unblocked system (2.1) are recalled from Kailath [1980] and Hespanha [2009] (page 178).

Definition 2.3.1. *The finite zeros of the transfer function $W(z) = C(zI - A)^{-1}B + D$ with minimal realization $\{A, B, C, D\}$ are defined to be the finite values of z for which the rank of the following system matrix falls below its normal rank*

$$M(z) = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}. \quad (2.15)$$

Further, $W(z)$ is said to have an infinite zero when $n + \text{rank}(D)$ is less than the normal rank of $M(z)$, or equivalently the rank of D is less than the normal rank of $W(z)$.

Similar to the above definition, the definition for zeros of the blocked system (2.4) is provided in the following.

Definition 2.3.2. *The finite zeros of the transfer function $W_b(Z) = C_b(ZI - A_b)^{-1}B_b + D_b$ with minimal realization $\{A_b, B_b, C_b, D_b\}$ are defined to be the finite values of Z for which the rank of the following system matrix falls below its normal rank*

$$M_b(Z) = \begin{bmatrix} ZI - A_b & -B_b \\ C_b & D_b \end{bmatrix}. \quad (2.16)$$

Further, $W_b(Z)$ is said to have an infinite zero when $n + \text{rank}(D_b)$ is less than the normal rank of $M_b(Z)$, or equivalently the rank of D_b is less than the normal rank of $W_b(Z)$.

Remark 2.3.3. Those zeros defined through the system matrix are referred to as *invariant zeros* in the literature Rosenbrock [1970]. There also exists the notion of *transmission zeros* which are obtainable from the Smith-McMillan form of a transfer function matrix. It is worthwhile noting that when a realization is minimal invariant zeros and transmission zeros coincide. However, when it is not minimal the invariant zeros includes the transmission zeros. Furthermore, all unreachable and unobservable modes known as *input decoupling zeros* and *output decoupling zeros* are also zeros¹. This thesis uses an algebraic approach to study zeros. The geometric approach has also been used to handle zeros see e.g. Wonham [1979], Morse [1973] and Grasselli and Longhi [1988].

2.3.2 Blocked Systems and Unblocked Systems - the Normal Rank

As shown in the last subsection, the normal rank plays an important role in the characterization of zeros. Thus, in this subsection an important result regarding the relationship between the normal rank of $W_b(Z)$ and the normal rank of $W(z)$ is given. A restrictive version of the following result is initially stated as Theorem 3 in Chen et al. [2012] only for linear time-invariant systems with $p \geq m$ and full-column normal rank. The next theorem extends this result to linear time-invariant systems with an arbitrary normal rank.

Theorem 2.3.4. Consider the unblocked transfer function $W(z)$ given by (2.2) and the blocked transfer function $W_b(Z)$ given (2.6). Then the following equality relates their normal rank.

$$\text{normal rank}(W_b(Z)) = N \times \text{normal rank}(W(z)).$$

Proof. To prove the conclusion of theorem, we modify the proof of Theorem 3 in Chen et al. [2012]. First let r and S denote the normal rank of $W(z)$ and the normal rank of $W_b(Z)$, respectively.

There exists a complex number $Z_0 \neq 0$ and N distinct complex numbers $z_i, i = 1, 2, \dots, N$ such that $\det(\mathcal{A}(Z_0)) \neq 0$, $\text{rank}(\mathcal{B}(Z_0)) = S$, $z_i^N = Z_0, i = 1, 2, \dots, N$ and $\text{rank}(P(z_i)) = r, i = 1, 2, \dots, N$. Define Y and Λ as in (2.13), then it follows from Lemma 2.2.2 that $\mathcal{B}(Z_0)Y = \Lambda$. Noting that $z_i \neq z_l$ for $i \neq l$, we see that Y is a nonsingular matrix because it is a Kronecker product of a **nonsingular** Vandermonde matrix with the identity matrix. Furthermore, Λ can be written as

$$\begin{aligned} \Lambda &= \begin{bmatrix} I_p & I_p & \cdots & I_p \\ z_1 I_p & z_2 I_p & \cdots & z_N I_p \\ \vdots & \ddots & \ddots & \vdots \\ z_1^{N-1} I_p & z_2^{N-1} I_p & \cdots & z_N^{N-1} I_p \end{bmatrix} \begin{bmatrix} P(z_1) & 0 & \cdots & 0 \\ 0 & P(z_2) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & P(z_N) \end{bmatrix} \\ &= \tilde{Y} \text{diag}\{P(z_1), P(z_2), \dots, P(z_N)\}. \end{aligned} \quad (2.17)$$

¹For more detailed study about different notions of zeros and their relationships, interested readers can refer to Macfarlane and Karcnias [1976], Rosenbrock [1970] and Grasselli and Longhi [1991b].

Observe that \tilde{Y} is a nonsingular matrix. Hence, Λ has rank Nr which implies that $\text{rank}(\mathcal{B}(Z_0)) = \text{rank}(\Lambda) = Nr$. This together with the fact that $\det(\mathcal{A}(Z_0)) \neq 0$ implies that $\text{rank}(\mathcal{V}(Z_0)) = \text{rank}(\mathcal{A}^{-1}(Z_0)\mathcal{B}(Z_0)) = Nr$. Since the rank of $W_b(Z)$ for a particular choice of Z is equal to Nr ; thus, the normal rank of $W_b(Z) \geq Nr$ i.e. $S \geq Nr$.

Conversely, there exists a complex number $Z_0 \neq 0$ such that $\det(\mathcal{A}(Z_0)) \neq 0$ and $\mathcal{B}(Z_0)$ has rank S . Now let $z_i, i = 1, 2, \dots, N$ be complex numbers such that $z_i^N = Z_0, i = 1, 2, \dots, N$ and $\text{rank}(P(z_i)) = \text{rank}(P(z_l)), z_i \neq z_l$. Define Y and Λ as in (2.13), then it follows from Lemma 2.2.2 that $\mathcal{B}(Z_0)Y = \Lambda$. Noting that $z_i \neq z_l$ for $i \neq l$ so, Y is nonsingular. Hence, Λ has rank S . Hence, it follows from the definition of Λ that all $P(z_i), i \in \{1, 2, \dots, N\}$ have the same rank equal to S/N , which must therefore be an integer. Since for particular z_i , we have $P(z_i)$ and thus $W(z_i)$ of rank S/N there holds normal rank $W(z) \geq S/N$ or $Nr \geq S$.

Now, by using the both inequalities i.e. $Nr \geq S$ and $S \geq Nr$, one can reason that $S = Nr$. \square

The above theorem relates the normal rank of associated unblocked and blocked transfer functions. The normal rank of associated system matrices to the respective transfer functions and to each other can also be easily related.

Lemma 2.3.5. *Consider the unblocked transfer function $W(z)$ given by (2.2) and its corresponding system matrix denoted by $M(z)$. Then the following equality holds.*

$$\text{normal rank}(M(z)) = n + \text{normal rank}(W(z)).$$

Proof.

$$M(z) = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ C(zI - A)^{-1} & I \end{bmatrix} \begin{bmatrix} zI - A & -B \\ 0 & W(z) \end{bmatrix}.$$

Observe that the normal rank $(M(z)) = \text{normal rank}(zI - A) + \text{normal rank}(W(z)) = n + \text{normal rank}(W(z))$. \square

Corollary 2.3.6. *The normal rank of $M(z)$ is $n + r$ if and only if the normal rank of $M_b(Z)$ is $n + Nr$.*

Proof. The proof is immediate using the results of Lemma 2.3.5 and Theorem 2.3.4. \square

2.3.3 Blocked Systems and Unblocked Systems - Zeros

In the last subsection, the relationship between the normal rank of $W_b(Z)$ and the normal rank of $W(z)$ was explored. In this subsection, the relationship between zeros of blocked systems and those of their corresponding unblocked systems is investigated. As stated earlier, due to the complexity of analysis, three cases are separately discussed, that is, 1) finite nonzero zeros; 2) zeros at infinity; and 3) zeros at the origin.

A sufficient condition under which the blocked system (2.4) has a finite nonzero zero has been provided in Bittanti and Colaneri [2009]. Later, in Chen et al. [2012] and Zamani et al. [2011], a necessary and sufficient condition has been provided where a *tall* blocked system with *full-column normal rank* has a finite nonzero zero. A necessary and sufficient condition for the blocked system (2.4) to have a finite nonzero zero without imposing any condition either on the structure of the blocked system or on the normal rank of its associated transfer function matrix is provided as the following theorem.

Theorem 2.3.7. *Consider the unblocked system (2.1) with transfer function $W(z)$ given by (2.2) and the blocked system (2.4) with transfer function $W_b(Z)$ given by (2.6). Suppose that the quadruple $\{A, B, C, D\}$ is minimal. Define $\omega \triangleq \exp(\frac{2\pi j}{N})$. It follows that*

1. *If $W(z)$ has a finite zero at $z_0 \neq 0$ then $W_b(Z)$ has a finite zero at $Z = Z_0 = z_0^N \neq 0$.*
2. *If $W_b(Z)$ has a finite zero at $Z = Z_0 \neq 0$ then, for any z_0 satisfying $z_0^N = Z_0$, $W(z)$ has a finite zero at one or more of $z = z_0 \neq 0$ or $z = \omega z_0 \neq 0 \dots z = \omega^{N-1} z_0 \neq 0$.*

Proof. Part one. Let r denote the normal rank of $W(z)$. Assume that the unblocked system has a zero at z_0 which implies that $\text{rank}(P(z_0)) < r$. Now, let $z_i = \omega^i z_0, i = 1, 2, \dots, N$, where $\omega = \exp(\frac{2\pi j}{N})$, be N distinct complex numbers and $z_i^N = z_0^N = Z_0, i = 1, 2, \dots, N$. One can define Y and Λ as in (2.13); then using Lemma 2.2.2 it is immediate that $\mathcal{B}(Z_0)Y = \Lambda$. Moreover, with the help of equation (2.17), we can obtain $\mathcal{B}(Z_0)Y = \tilde{Y} \text{diag}\{P(z_1), P(z_2), \dots, P(z_N)\}$. Since z_i are chosen to be distinct Y and \tilde{Y} are nonsingular matrices and $\text{rank}(\mathcal{B}(Z_0)) = \text{rank}(\text{diag}\{P(z_1), P(z_2), \dots, P(z_N)\})$. Since $\text{rank}(P(z_i)) \leq r, i = 1, 2, \dots, N-1$, the assumption that $\text{rank}(P(z_0)) < r$ implies that $\text{rank}(\mathcal{B}(Z_0)) < Nr$ so, $\text{rank}(V(Z_0)) < Nr$. Furthermore, from Theorem 2.3.4, it is known that $\text{normal rank}(W_b(Z)) = Nr$; hence, the blocked system (2.4) has a finite zero at $Z_0 = z_0^N$.

Part two. Now in part two, suppose that Z_0 is a zero for the system matrix of (2.4). Also, note that there exist N distinct complex numbers $z_i, i \in \{1, 2, \dots, N\}$ such that $z_i^N = Z_0, i \in \{1, 2, \dots, N\}$. Hence, according to the result of Lemma 2.2.2, Y is a nonsingular matrix and $\text{rank}(\mathcal{B}(Z_0)) = \text{rank}(\Lambda)$. Since $\text{rank}(\Lambda)$ is less than normal rank, one or more of $P(z_i), i \in \{1, 2, \dots, N\}$ should have less than the normal rank. The latter implies that the system (2.1) has a finite zero at one or more of $z = z_0 \neq 0$ or $z = \omega z_0 \neq 0 \dots z = \omega^{N-1} z_0 \neq 0$. □

Remark 2.3.8. *It is worthwhile mentioning that the factorization recalled in Remark 1 of Chen et al. [2012], can also be used to prove the statement of the above theorem.*

The above theorem treats zeros of the blocked system for choice of finite nonzero zeros; it is natural to ask what happens to zeros at infinity, and the following theorem deals with this case.

Theorem 2.3.9. Consider the unblocked system (2.1) with transfer function $W(z)$ given by (2.2) and the blocked system (2.4) with transfer function $W_b(Z)$ given by (2.6). Suppose that the quadruple $\{A, B, C, D\}$ is minimal. Then $W(z)$ has a zero at $z = \infty$ if and only if $W_b(Z)$ has a zero at $Z = \infty$.

Proof. Sufficiency. To prove the sufficiency part, first the rank of D_b and the rank of D (here the case where $N = 2$ is discussed and generalization to an arbitrary N is straightforward) are related. Suppose that D has rank q , then there exist invertible matrices \bar{T} and \bar{S} such that $\bar{S}D\bar{T} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}$. Then one can write

$$\begin{bmatrix} \bar{S} & 0 \\ 0 & \bar{S} \end{bmatrix} \begin{bmatrix} D & 0 \\ CB & D \end{bmatrix} \begin{bmatrix} \bar{T} & 0 \\ 0 & \bar{T} \end{bmatrix} = \begin{bmatrix} \bar{S}D\bar{T} & 0 \\ \bar{S}CB\bar{T} & \bar{S}D\bar{T} \end{bmatrix}.$$

It now becomes immediate that $\text{rank}(D_b) \geq 2q$ i.e. $\text{rank}(D_b) \geq 2\text{rank}(D)$, and in a general case with an arbitrary N one obtains $\text{rank}(D_b) \geq N\text{rank}(D)$.

Now suppose that $W_b(Z)$ has a zero at infinity and let r denote the normal rank of $W(z)$, then according to the Definition 2.3.2, the matrix $\text{rank}(D_b) < Nr$. Now, using the result of Theorem 2.3.4 one can write $\text{rank}(D_b) < N \times \text{normal rank}(W(z))$. Then it becomes immediate that $\text{rank}(D) < r$ which implies that $W(z)$ has a zero at infinity.

Necessity. The proof of necessity part is immediate by using the structure of D_b . \square

Finally in the remainder of this subsection, zeros of blocked systems are examined for the choice of zeros at the origin. In order to deal with this case, the following Lemma is adopted from Varga and Van Dooren [2003]. It is obtained by specializing Lemma 1 of Varga and Van Dooren [2003] to the case where the unblocked system is time-invariant.

Lemma 2.3.10. Varga and Van Dooren [2003]

Let $\tilde{A}_b = I_N \otimes A$, $\tilde{B}_b = I_N \otimes B$, $\tilde{C}_b = I_N \otimes C$ and $\tilde{D}_b = I_N \otimes D$. Furthermore, define $E_Z \triangleq \begin{bmatrix} 0 & 1 & 0 \\ 0 & & \ddots \\ \vdots & & \ddots & 1 \\ Z & 0 & 0 \end{bmatrix}$, $E_Z \in \mathbb{C}^{N \times N}$ and $\tilde{E}_Z = E_Z \otimes I_n$ and \otimes denotes the Kronecker product. Then there exist invertible matrices T_l and T_r and matrices X and Y such that for all $Z \in \mathbb{C}$

$$\begin{bmatrix} I_{n(N-1)} & 0 & 0 \\ 0 & ZI - A_b & -B_b \\ 0 & C_b & D_b \end{bmatrix} = \begin{bmatrix} T_l & 0 \\ X & I \end{bmatrix} \begin{bmatrix} \tilde{E}_Z - \tilde{A}_b & -\tilde{B}_b \\ \tilde{C}_b & \tilde{D}_b \end{bmatrix} \begin{bmatrix} T_r & Y \\ 0 & I \end{bmatrix}. \quad (2.18)$$

The preceding result enables us to treat zeros at the origin. The following theorem studies this type of zero.

Theorem 2.3.11. Consider the unblocked system (2.1) with transfer function $W(z)$ given by (2.2) and the blocked system (2.4) with transfer function $W_b(Z)$ given by (2.6). Suppose that the quadruple $\{A, B, C, D\}$ is minimal. Then $W(z)$ has a zero at $z = 0$ if and only if $W_b(Z)$ has a zero at $Z = 0$.

Proof. Sufficiency. Let $W(z)$ have normal rank r so, $W_b(Z)$ has normal rank Nr and $M_b(Z)$ has normal rank $n + Nr$. Now suppose that $W_b(Z)$ has a zero at $Z = 0$, so that

$M_b(Z)$ has rank less than its normal rank at $Z = 0$. Then $\text{rank} \left(\begin{bmatrix} I_{n(N-1)} & 0 & 0 \\ 0 & -A_b & -B_b \\ 0 & C_b & D_b \end{bmatrix} \right) < N(n + r)$. Now by using the conclusion of the Lemma 2.3.10, one can easily write

$$\begin{bmatrix} I_{n(N-1)} & 0 & 0 \\ 0 & -A_b & -B_b \\ 0 & C_b & D_b \end{bmatrix} = \begin{bmatrix} T_l & 0 \\ X & I \end{bmatrix} \begin{bmatrix} \tilde{E}_0 - \tilde{A}_b & -\tilde{B}_b \\ \tilde{C}_b & \tilde{D}_b \end{bmatrix} \begin{bmatrix} T_r & Y \\ 0 & I \end{bmatrix} \quad (2.19)$$

and so $\text{rank} \left(\begin{bmatrix} \tilde{E}_0 - \tilde{A}_b & -\tilde{B}_b \\ \tilde{C}_b & \tilde{D}_b \end{bmatrix} \right) < N(n + r)$.

On the other hand, $\text{rank} \left(\begin{bmatrix} \tilde{E}_0 - \tilde{A}_b & -\tilde{B}_b \\ \tilde{C}_b & \tilde{D}_b \end{bmatrix} \right) \geq N \text{rank} \left(\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} \right)$. (This follows from the fact that with row and column reordering the matrix on the left can be made upper triangular with diagonal blocks all of the form $\begin{bmatrix} -A & -B \\ C & D \end{bmatrix}$). The last two inequalities imply that $\text{rank} \left(\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} \right) < n + r$ and so the unblocked system has a zero at $z = 0$.

Necessity. Suppose $W(z)$ has a zero at $z = 0$. Note that the associated system is finite dimensional and therefore A has only a finite number of eigenvalues. Thus there exists $\rho > 0$ such that $A - zI$ is invertible for all real numbers z with $0 < z < \rho$. Let ϵ be any such number. (ϵ is further restricted subsequently). Let $A_\epsilon \triangleq A - \epsilon I$; hence, A_ϵ is nonsingular. Clearly, $z = 0$ is a point where there is rank loss of the system matrix associated with the quadruple $\{A, B, C, D\}$ if and only if $z = \epsilon$ is a point where there is rank loss of the system matrix associated with the quadruple $\{A_\epsilon, B, C, D\}$. Using the result of the Theorem 2.3.7, one can obtain that ϵ^N is a point where there is rank loss of the system matrix associated with a quadruple $\{\bar{A}_{b_\epsilon}, \bar{B}_{b_\epsilon}, \bar{C}_{b_\epsilon}, \bar{D}_{b_\epsilon}\}$ (where the quadruple $\{\bar{A}_{b_\epsilon}, \bar{B}_{b_\epsilon}, \bar{C}_{b_\epsilon}, \bar{D}_{b_\epsilon}\}$ characterizes the blocked system associated with the quadruple $\{A_\epsilon, B, C, D\}$). By hypothesis, the following inequality holds.

$$\text{rank} \left(\begin{bmatrix} \epsilon^N I - \bar{A}_{b_\epsilon} & \bar{B}_{b_\epsilon} \\ \bar{C}_{b_\epsilon} & \bar{D}_{b_\epsilon} \end{bmatrix} \right) < \text{normal rank} \left(\begin{bmatrix} ZI - \bar{A}_{b_\epsilon} & \bar{B}_{b_\epsilon} \\ \bar{C}_{b_\epsilon} & \bar{D}_{b_\epsilon} \end{bmatrix} \right). \quad (2.20)$$

With the help of equation (2.18), one can write

$$n(N-1) + \text{normal rank} \left(\begin{bmatrix} ZI - A_b & -B_b \\ C_b & D_b \end{bmatrix} \right) = \text{normal rank} \left(\begin{bmatrix} \tilde{E}_Z - \tilde{A}_b & -\tilde{B}_b \\ \tilde{C}_b & \tilde{D}_b \end{bmatrix} \right)$$

and

$$n(N-1) + \text{normal rank} \left(\begin{bmatrix} ZI - \bar{A}_{b_\epsilon} & \bar{B}_{b_\epsilon} \\ \bar{C}_{b_\epsilon} & \bar{D}_{b_\epsilon} \end{bmatrix} \right) = \text{normal rank} \left(\begin{bmatrix} \tilde{E}_Z - \tilde{A}_{b_\epsilon} & -\tilde{B}_b \\ \tilde{C}_b & \tilde{D}_b \end{bmatrix} \right),$$

where $\tilde{A}_{b_\epsilon} = I_N \otimes A_\epsilon$. It is proved in the following that the equality below holds.

$$\text{normal rank} \left(\begin{bmatrix} ZI - \bar{A}_{b_\epsilon} & \bar{B}_{b_\epsilon} \\ \bar{C}_{b_\epsilon} & \bar{D}_{b_\epsilon} \end{bmatrix} \right) = \text{normal rank} \left(\begin{bmatrix} ZI - A_b & -B_b \\ C_b & D_b \end{bmatrix} \right). \quad (2.21)$$

Suppose that $W(z) = D + C(zI - A)^{-1}B$ and $\text{normal rank}(W(z)) = r$, $r \leq \min\{m, p\}$, then $W_\epsilon(z) \triangleq D + C(zI - A_\epsilon)^{-1}B = D + C((z + \epsilon)I - A)^{-1}B$ has the same normal rank i.e. r . By using the result of Theorem 2.3.4, one obtains $\text{normal rank}(W_b(Z)) = Nr$ and $\text{normal rank}(W_{b_\epsilon}(Z)) = Nr$ (where $W_{b_\epsilon}(Z)$ is the transfer function associated the quadruple $\{\bar{A}_{b_\epsilon}, \bar{B}_{b_\epsilon}, \bar{C}_{b_\epsilon}, \bar{D}_{b_\epsilon}\}$). Thus, with the help of Lemma 2.3.5, the equality (2.21) becomes immediate.

At this stage of proof, the following inequality is proved.

$$\text{rank}(M_b(0)) \leq \text{rank}(M_{b_\epsilon}(\epsilon^N)), \quad (2.22)$$

$$\text{where } M_{b_\epsilon}(\epsilon^N) = \begin{bmatrix} \epsilon^N I - \bar{A}_{b_\epsilon} & \bar{B}_{b_\epsilon} \\ \bar{C}_{b_\epsilon} & \bar{D}_{b_\epsilon} \end{bmatrix}.$$

Suppose that $\text{rank} \left(\begin{bmatrix} -A^N & B_b \\ C_b & D_b \end{bmatrix} \right) = q$; then there exists a $q \times q$ submatrix of $M_b(0)$ such that its determinant is nonzero. Recall that the determinant function is a continuous function of the entries of the associated matrix and the quadruple $\{\bar{A}_{b_\epsilon}, \bar{B}_{b_\epsilon}, \bar{C}_{b_\epsilon}, \bar{D}_{b_\epsilon}\}$ can be obtained from $\{A_b, B_b, C_b, D_b\}$ with a perturbation ϵ of the underlying unblocked matrix A . Therefore, since ρ can be chosen arbitrarily small while positive, one may assume that for any positive ϵ with $\epsilon < \rho$, there exists a $q \times q$ submatrix of $M_{b_\epsilon}(\epsilon^N)$ which has nonzero determinant, so that $\text{rank} \left(\begin{bmatrix} \epsilon^N I - \bar{A}_{b_\epsilon} & \bar{B}_{b_\epsilon} \\ \bar{C}_{b_\epsilon} & \bar{D}_{b_\epsilon} \end{bmatrix} \right) \geq q$.

Finally, by combining (2.20), (2.21) and (2.22), the following inequality readily follows.

$$\text{rank} \left(\begin{bmatrix} -A^N & B_b \\ C_b & D_b \end{bmatrix} \right) < \text{normal rank} \left(\begin{bmatrix} ZI - A_b & -B_b \\ C_b & D_b \end{bmatrix} \right).$$

Equivalently, $Z = 0$ is a zero for $W_b(Z)$. □

In this subsection, zeros of the blocked system for the whole complex plane including infinity have been studied. One can naturally ask what are the zero prop-

erties of the blocked system for a generic choice of parameter matrices A, B , etc. It is important to recall that zeros of the unblocked system (2.1) for a generic choice of parameter matrices have been studied in the literature, see Kailath [1980], Filler [2010] and Karcanias and Kouvaritakis [1979]. However, since the parameter matrices of the blocked system (2.4) are structured and their entries cannot independently assume generic values, the study of zeros in this case is not trivial.

2.3.4 Zeros Properties of Blocked Systems Under a Generic Setting

The following theorem investigates the zeros of the system (2.4) for a generic choice of system matrices A, B, C, D . It extends the earlier result of Zamani et al. [2011], which only considered tall blocked systems. Zamani et al. [2011] states that tall blocked systems have no zeros when matrices A, B, C, D assume generic values. The following theorem generalizes the result in Zamani et al. [2011] by specifying that nonsquare blocked systems have no zeros when matrices A, B, C, D accept generic values and square blocked systems generically only have finite zeros and these finite zeros have one-dimensional kernel.

Theorem 2.3.12. *Consider the system (2.1) defined by the quadruple $\{A, B, C, D\}$, in which the individual matrices are generic and m, n, p are nonzero. Then*

1. *If $p > m$, the system matrix of the blocked system has full-column rank for all Z .*
2. *If $p < m$, the system matrix of the blocked system has full-row rank for all Z .*
3. *If $p = m$, then the system matrix of the blocked system must have finite zeros with one-dimensional kernel.*

Proof. We first suppose that $p > m$. Then it can be readily shown that the system matrix of tall unblocked systems generically have full-column normal rank. Furthermore, Filler [2010] showed that tall unblocked systems are generically zero-free. If the blocked system had its system matrix with less than full-column rank for a finite $Z_0 \neq 0$, then according to Theorem 2.3.7, there would be necessarily a nonzero nullvector of the system matrix of the unblocked system for $z_0 \neq 0$ equal to some $N - th$ root of Z_0 , which would be a contradiction. If the blocked system had a zero at $Z_0 = \infty$, then based on Theorem 2.3.9 the D matrix of the unblocked system would be less than full-column rank which would be a contradiction. Analogously, using the argument in Theorem 2.3.11, one can easily conclude that a blocked system has full-column rank system matrix at $Z_0 = 0$. The case $p < m$ can be done similarly. One can appeal to the fact, demonstrated in Bittanti and Colaneri [2009] page 180, that if $W_b(Z)$ is the blocked transfer function associated with $W(z)$, then $P_1 W_b^T(Z) P_2$ for certain permutation matrices P_1, P_2 is the blocked transfer function associated with $W^T(z)$.

Now we consider the case $p = m$; since D is generic, it has full-column rank. Hence, based on the conclusion of Theorem 2.3.9, both the unblocked system and the blocked system do not have zeros at infinity. In the second part of this proof,

we use the conclusion of Theorem 2.3.7. Furthermore, one should note that since matrices A, B, C and D assume generic values it can be easily understood that the quadruple $\{A_b, B_b, C_b, D_b\}$ is a minimal realization. Now, based on the fact that D_b is nonsingular, one can conclude that the zeros of the blocked system are the eigenvalues of $A_b - B_b D_b^{-1} C_b$. If the eigenvalues of $A_b - B_b D_b^{-1} C_b$ are distinct, then the associated eigenspace for each eigenvalue is one-dimensional; it is equivalent to saying that the associated kernel of $M_b(Z)$ evaluated at the eigenvalue has dimension one. One should note that the unblocked system has distinct zeros due to the genericity assumption. Furthermore, zeros of the unblocked system generically have distinct magnitudes except for complex conjugate pairs. It is obvious that those zeros of the unblocked system with distinct magnitudes produce distinct blocked zeros. Now, we focus on zeros of the unblocked system with the same magnitudes, i.e. complex conjugate pairs. The only case where the generic unblocked system has distinct zeros but its corresponding blocked system has non-distinct zeros happens when the $N - th$ power of the complex conjugate zeros of the unblocked system coincide. We now show by contradiction that this is generically impossible. In order to illustrate a contradiction, suppose that the unblocked system has a complex conjugate pair, say z_{01} and z_{01}^* . If they produce an identical zero for the blocked system, their $N - th$ powers must be the same. This condition implies that the angle between z_{01} and z_{01}^* has to be exactly $\frac{2\pi h}{N}$, where h is an integer, which contradicts the genericity assumption for the unblocked system. Hence, the zeros of the blocked system generically have distinct values and consequently the corresponding kernels of system matrix evaluated at the zeros are one-dimensional. \square

2.4 Summary

Zeros of blocked systems resulting from blocking of linear time-invariant systems were explored in this chapter. It was demonstrated that the blocked system has a zero if and only if the associated unblocked linear time-invariant system has a zero. Moreover, nonsquare blocked systems are generically zero-free; however, square blocked systems have only finite zeros and the kernel associated with each individual zero is of dimension one.

On the Normal rank and Zero-freeness of Tall Multirate Systems with Fast Outputs at the Fundamental Rate

Abstract

In this chapter, zeros of tall discrete-time linear systems with multirate outputs are examined. In the literature, zeros of multirate systems are defined as those of their corresponding blocked time-invariant systems; hence, the focus is on zeros of blocked systems obtained from blocking of linear systems with multirate outputs. We examine zeros of blocked systems where the parameter matrices of the associated unblocked systems accept generic values. To this end, we need to accurately calculate the normal rank of the associated blocked system matrix. Then under the above setup, it is shown that the blocked systems have no zeros apart from zeros possibly at the origin and at infinity. The occurrence of zeros at the origin and infinity is more complicated and depends on the blocking delay and the input, state and output dimensions.

3.1 Introduction

The main motivation for us to study multirate systems was completely covered in Chapter 1. In the applications which we are interested in; systems usually have more outputs than inputs Deistler et al. [2010b] and Raknerud et al. [2007] i.e. tall systems. To the author's best knowledge, tall and very tall linear multi-rate systems have not been studied in great depth. This chapter however does try to formulate some general properties of tall multi-rate systems. Consequently, the focus is not on a particular application problem but rather on a bigger framework, which is the system theoretical issues associated with such systems.

In the single-rate scenario, Filler [2010] showed that tall linear time-invariant systems are generically zero-free. This means that the set of zero-free systems is a

generic (an open and dense) subset of the parameter space. Deistler et al. [2010b] used this property and illustrated that systems parameters can be identified through linear calculations from the observed data, using *Yule-Walker* equations Lütkepohl [2005]. A corresponding demonstration until now has been lacking for the multirate case, and the central task of this chapter is to address that shortcoming. Specifically, this chapter deals with zeros of multirate linear systems, when the parameter matrices A, B, C and D corresponding to a minimal state space realization of the underlying transfer function matrix assume generic values. It is shown when these systems are zero-free and when they have zeros at the origin and infinity.

While the prime motivation for this chapter is to demonstrate a property which implies, as noted above, substantial simplification in the identification or modeling task, it is worthwhile mentioning that the result may have separate importance from a control design perspective; zeros which are unstable or stable but close to a stability boundary can provide obstructions to the existence of inverses of linear systems and more generally, the design of high performance controllers. The results of this chapter suggest that, when one is dealing with a generic system, the controller design may then be easier if one can add extra sensors to make the system have more outputs than inputs, and thereby suppress occurrence of any zeros at all, apart possibly from zeros at zero or infinity.

In the systems and control literature, the zeros of multirate systems, at a certain point of time, are defined as those of their corresponding blocked systems Bittanti and Colaneri [2009]. To the best of the author's knowledge, there are few works on zeros of multirate systems. Among such works one should mention Bolzern et al. [1986], Grasselli and Longhi [1988], Bittanti and Colaneri [2009], Zamani et al. [2011], Chen et al. [2012], Zamani and Anderson [2012], Zamani et al. [2013a] and Anderson et al. [2013a]. These references were widely reviewed in Chapter 1.

The main objective of this chapter is to investigate zeros of tall blocked systems resulting from blocking of a multirate linear system with a generic choice of parameter matrices appearing in the state space description of the system. The results of this study reveal what kind of zeros tall blocked systems have for almost all choices of parameter matrices. Note that there are already some results in the literature dealing with zeros of unblocked tall linear time-invariant (LTI) systems with generic parameter matrices Anderson and Deistler [2008], Filler [2010], Wonham [1979] and Kailath [1980]. However, there has been a gap in the literature regarding the study of blocked systems formed by the blocking of multirate linear systems; this process results in a time-invariant system with relations among the entries of the state-variable matrices of the blocked system, i.e. so that the state-variable matrices are not fully generic.

Since the analysis of zeros for tall blocked systems is quite involved; hence, three separate cases are considered, that is, 1) finite nonzero system zeros; 2) system zeros at infinity; and 3) system zeros at zero. The next section focuses on zeros of tall blocked systems associated with finite nonzero zeros. It is explicitly established that tall blocked systems generically have no finite nonzero zeros. As a byproduct in this section, the generic rank of a system matrix resulting from blocking a multirate system is precisely calculated. Following this, in Section 3.3 zeros of tall blocked

systems are examined at $Z = 0$ and $Z = \infty$. It is shown when tall blocked systems can have a zero at $Z = 0$ or $Z = \infty$ and when they are zero-free at those aforementioned points. Finally, Section 3.4 offers concluding remarks.

3.2 Blocked Systems with Generic Parameters - Finite Nonzero Zeros

In this section, first the formulation of the problem under study is introduced. Then attention is given to the analysis of zeros for tall blocked systems with generic parameters, only finite nonzero zeros are considered in this section. In the next section, infinite zeros and zeros at the origin are explored.

The dynamics of an underlying system operating at the highest sample rate are defined by

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, \\ y_t &= Cx_t + Du_t, \end{aligned} \tag{3.1}$$

where $t \in \mathbb{Z}$, $x_t \in \mathbb{R}^n$ is the state, $y_t \in \mathbb{R}^p$ is the output, and $u_t \in \mathbb{R}^m$ is the input. For this system, y_t exists for all t , and, separately, can be measured at every time t . However, we are also interested in the situation, where though y_t exists for all t , not every entry is measured for all t . In particular, this chapter considers the case where y_t has components that are observed at different rates. For simplicity, a case where outputs are provided at two rates which we refer to as the fast rate and the slow rate is discussed here.

Without loss of generality, y_t can be decomposed as $y_t = \begin{bmatrix} y_t^f & y_t^s \end{bmatrix}^T$ where $y_t^f \in \mathbb{R}^{p_f}$ is associated with the fast rate and observed at all t , and $y_t^s \in \mathbb{R}^{p_s}$ is associated with slow rate and observed at $t = 0, N, 2N, \dots$, also $p_f > 0, p_s > 0$ and $p_f + p_s = p$. Accordingly, the matrices C and D can be also expressed as

$$C = \begin{bmatrix} C^f \\ C^s \end{bmatrix}, D = \begin{bmatrix} D^f \\ D^s \end{bmatrix}.$$

Thus, the multirate linear system corresponding to what is measured has the following dynamics

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, \quad t = 0, 1, 2, \dots, \\ y_t^f &= C^f x_t + D^f u_t, \quad t = 0, 1, 2, \dots, \\ y_t^s &= C^s x_t + D^s u_t, \quad t = 0, N, 2N, \dots \end{aligned} \tag{3.2}$$

We have N distinct alternative ways to block the system, depending on how the fast rates are grouped with the slow rates. Even though these N different systems share some common poles, their zeros are not identical in the whole complex plane (see Bittanti and Colaneri [2009], pages 173-179).

We index these systems with an integer value, called the *tag point*, $\tau \in \{1, 2, \dots, N\}$, and define

$$U_t^\tau \triangleq \begin{bmatrix} u_{t+\tau} \\ u_{t+\tau+1} \\ \vdots \\ u_{t+\tau+N-1} \end{bmatrix}, \quad Y_t^\tau \triangleq \begin{bmatrix} y_{t+\tau}^f \\ y_{t+\tau+1}^f \\ \vdots \\ y_{t+\tau+N-1}^f \\ y_{t+N}^s \end{bmatrix}, \quad (3.3)$$

$x_t^\tau \triangleq x_{t+\tau}$, where $t = 0, N, 2N, \dots$

Then the blocked system Σ_τ is defined by

$$\begin{aligned} x_{t+N}^\tau &= A_\tau x_t^\tau + B_\tau U_t^\tau, \\ Y_t^\tau &= C^\tau x_t^\tau + D^\tau U_t^\tau, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} A_\tau &\triangleq A^N, \quad B_\tau \triangleq \begin{bmatrix} A^{N-1}B & A^{N-2}B & \dots & AB & B \end{bmatrix}, \\ C_\tau &\triangleq \begin{bmatrix} C^f & A^T C^f & \dots & A^{(N-1)T} C^f & A^{(N-\tau)T} C^s \end{bmatrix}^T, \\ D_\tau &\triangleq \begin{bmatrix} D^f & 0 & \dots & 0 \\ C^f B & D^f & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-2}B & C^f A^{N-3}B & \dots & D^f \\ \hline & D_\tau^s & & \end{bmatrix}, \end{aligned} \quad (3.5)$$

where $D_\tau^s = [C^s A^{N-\tau-1}B \dots C^s B D^s 0 \dots 0]$ for $\tau < N$ with $\tau - 1$ zero blocks of size $p_s \times m$, and when $\tau = N$, it is given by $D_N^s = [D^s 0 \dots 0]$ where there are $N - 1$ zero blocks of size $p_s \times m$.

Reference Bittanti and Colaneri [2009] defines zeros of (3.2) at time τ as zeros of its corresponding blocked system Σ_τ ¹. Hence, in the rest of this section, we focus on zeros of the blocked system $\Sigma_\tau \forall \tau \in \{1, 2, \dots, N\}$.

Let $M_\tau(Z)$ denote the system matrix corresponding to the systems Σ_τ . Here, we follow the same definition of zeros as in Chapter 2. We also provide the next definition for the geometric multiplicity of a zero as it is needed in the upcoming sections.

Definition 3.2.1. *The geometric multiplicity of a finite zero $Z_0 \in \mathbb{C}$ is normal rank of $M_\tau(Z)$ -rank($M_\tau(Z_0)$). Moreover, the geometric multiplicity of a zero at infinity is normal rank of $M_\tau(Z) - n - \text{rank}(D_\tau)$.*

In this chapter, the term multiplicity is used to refer to the geometric multiplicity. Zeros of $\Sigma_\tau \forall \tau \in \{1, 2, \dots, N\}$ under a genericity assumption on the matrices of the

¹Zeros of the transfer function defined from (3.4) are identical with those defined here, provided the quadruple $\{A_\tau, B_\tau, C_\tau, D_\tau\}$ is minimal.

unblocked system and a tallness assumption are treated in this chapter. Given that $p_f, p_s > 0$, it proves convenient to consider a partition of the set of possible values of p_f and p_s defining tallness of the blocked transfer function into two subsets, as follows.

1. $p_f > m$.
2. $p_f \leq m, Np_f + p_s > Nm$.

The first case is common, perhaps even overwhelmingly common, in econometric modeling but the second case is important from a theoretical point of view, and possibly in other applications. Results in the current chapter are able to cover both cases, but separate treatment is required.

3.2.1 Case $p_f > m$

According to Definition 2.3.2, the normal rank for the system matrix of $\sum_\tau \forall \tau \in \{1, 2, \dots, N\}$, plays an important role in the analysis of its zeros; thus, the following straightforward and preliminary result for the normal rank of $\sum_\tau \forall \tau \in \{1, 2, \dots, N\}$, is provided.

Lemma 3.2.2. *For generic choice of the matrices $\{A, B, C^s, C^f, D^f, D^s\}$, $p_f \geq m$, the system matrix of $\sum_\tau \forall \tau \in \{1, 2, \dots, N\}$, has normal rank of $n + Nm$.*

Proof. In a generic setting and with $p_f \geq m$, the matrix D^f is of full-column rank. So, due to the structure of $D_\tau \forall \tau \in \{1, 2, \dots, N\}$, one can easily conclude that $D_\tau \forall \tau \in \{1, 2, \dots, N\}$, is of full-column rank as well. Furthermore,

$$\begin{aligned} M_\tau(Z) &= \begin{bmatrix} ZI - A_\tau & -B_\tau \\ C_\tau & D_\tau \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ C_\tau(ZI - A_\tau)^{-1} & I \end{bmatrix} \begin{bmatrix} ZI - A_\tau & -B_\tau \\ 0 & C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau \end{bmatrix}. \end{aligned} \quad (3.6)$$

Now observe that $M_\tau(Z)$ has $n + Nm$ columns so, $n + Nm \geq \text{normal rank}(M_\tau(Z)) = \text{normal rank}(ZI - A_\tau) + \text{normal rank}(C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau) \geq n + \text{rank}(\lim_{Z \rightarrow \infty} [C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau]) = n + \text{rank}(D_\tau) = n + Nm$. Hence, the normal rank of $M_\tau(Z)$ equals the number of its columns. \square

In the situation where $p_f > m$, obtaining a result on the absence of finite nonzero zeros is now rather trivial, since the blocked system contains a subsystem obtained by deleting some outputs which is provably zero-free.

Theorem 3.2.3. *For a generic choice of the matrices $\{A, B, C^s, C^f, D^s, D^f\}$, $p_f > m$, the system matrix of $\sum_\tau \forall \tau \in \{1, 2, \dots, N\}$, has full-column rank for all finite nonzero Z .*

Proof. Define a system matrix $M^f(Z)$ by deleting those rows of $M_\tau(Z)$, $\tau \in \{1, 2, \dots, N\}$, which contain any entries of C^s . Thus $M^f(Z)$ is a system matrix associated with

a blocked version of the original system with slow outputs completely discarded, i.e. of a time-invariant and not just a periodic system. With $p_f > m$, it was shown in Zamani et al. [2011] that $M^f(Z)$ is generically of full-column rank for all finite nonzero Z . Then it is immediate that $M_\tau(Z)$, $\tau \in \{1, 2, \dots, N\}$, will be of full-column rank for all finite nonzero Z . \square

3.2.2 Case $p_f \leq m$, $Np_f + p_s > Nm$

In the previous subsection, the case $p_f > m$ was treated where only considering the fast outputs alone generically leads to a zero-free blocked system, and the zero-free property is not disturbed by the presence of the further slow outputs. A different way in which the blocked system will be tall arises when $p_f \leq m$ and $Np_f + p_s > Nm$. The main result of this subsection is to show that $\sum_\tau \forall \tau \in \{1, 2, \dots, N\}$ with $p_f \leq m$, $Np_f + p_s > Nm$ is again generically zero-free. This case is harder to treat; in the conference paper Zamani and Anderson [2012], we treated the case under a restrictive assumption, namely that the system matrix of the blocked system had full-column rank, and we shall drop this assumption here. That the system matrix of the blocked system may indeed have less than full-column rank, so the extension is warranted, is exhibited in the following example.

Example 3.2.4. Consider a tall multi-rate system with $n = 1$, $m = 3$, $N = 2$, $p_f = 1$, $p_s = 5$. Let the parameter matrices for the multirate system be $A = a$, $B = [b_1 \ b_2 \ b_3]$, $C^f = c^f$, $C^s = [c_1^s \ c_2^s \ c_3^s \ c_4^s \ c_5^s]^T$, $D^f = [d_1^f \ d_2^f \ d_3^f]$ and

$$D^s = \begin{bmatrix} d_{11}^s & d_{12}^s & d_{13}^s \\ \vdots & \vdots & \vdots \\ d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}.$$

All the scalar parameters are generic. We consider $\tau = 1$ and write the associated system matrix as

$$M_1(Z) = \begin{bmatrix} Z - a^2 & -ab_1 & -ab_2 & -ab_3 & -b_1 & -b_2 & -b_3 \\ c^f & d_1^f & d_2^f & d_3^f & 0 & 0 & 0 \\ c^f a & c^f b_1 & c^f b_2 & c^f b_3 & d_1^f & d_2^f & d_3^f \\ c_1^s a & c_1^s b_1 & c_1^s b_2 & c_1^s b_3 & d_{11}^s & d_{12}^s & d_{13}^s \\ c_2^s a & c_2^s b_1 & c_2^s b_2 & c_2^s b_3 & d_{21}^s & d_{22}^s & d_{23}^s \\ c_3^s a & c_3^s b_1 & c_3^s b_2 & c_3^s b_3 & d_{31}^s & d_{32}^s & d_{33}^s \\ c_4^s a & c_4^s b_1 & c_4^s b_2 & c_4^s b_3 & d_{41}^s & d_{42}^s & d_{43}^s \\ c_5^s a & c_5^s b_1 & c_5^s b_2 & c_5^s b_3 & d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}.$$

It is obvious that first the two rows are (generically) linearly independent. Now consider rows

from 3 to 8; they can be written as a product of matrices $G\bar{\Gamma}$, with

$$G \triangleq \begin{bmatrix} c^f & c^f & c^f & c^f & d_1^f & d_2^f & d_3^f \\ c_1^s & c_1^s & c_1^s & c_1^s & d_{11}^s & d_{12}^s & d_{13}^s \\ c_2^s & c_2^s & c_2^s & c_2^s & d_{21}^s & d_{22}^s & d_{23}^s \\ c_3^s & c_3^s & c_3^s & c_3^s & d_{31}^s & d_{32}^s & d_{33}^s \\ c_4^s & c_4^s & c_4^s & c_4^s & d_{41}^s & d_{42}^s & d_{43}^s \\ c_5^s & c_5^s & c_5^s & c_5^s & d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}$$

and $\bar{\Gamma} \triangleq \text{diag}\{a, b_1, b_2, b_3, I_3\}$. The matrix G has rank at most 4; hence, with generic parameter matrices the normal rank of $M_1(Z)$ equals 6 and thus $M_1(Z)$ cannot attain full-column normal rank.

In the next part of this subsection, first the normal rank of $M_\tau(z)$ is characterized, and then the question of zero existence is discussed.

Proposition 3.2.5. Consider the system $\sum_\tau \forall \tau \in \{1, 2, \dots, N\}$, with $p_f \leq m$, $Np_f + p_s > Nm$ and generic values of the defining matrices $\{A, B, C^f, C^s, D^f, D^s\}$. Then

1. if $n \leq (N - \tau)(m - p_f)$, the matrix D_τ has rank equal to $(N - 1)p_f + m + n$;
2. if $n > (N - \tau)(m - p_f)$, the matrix D_τ has rank equal to $(\tau - 1)p_f + (N - \tau + 1)m$.

Proof. Refer to Appendix included at the end of this chapter for the complete proof. \square

It was shown that for $p_f > m$ the system matrix associated with the system \sum_τ has always full-column normal rank. However, for $p \leq m$ depending on the state space dimension, the system matrix may have less than full-column normal rank. The normal rank of $M_\tau(z)$ is precisely calculated in the following theorem.

Theorem 3.2.6. Consider the system $\sum_\tau, \tau \in \{1, 2, \dots, N\}$, with $p_f < m$, $Np_f + p_s > Nm$ and generic values of the defining matrices $\{A, B, C^f, C^s, D^f, D^s\}$. Then the normal rank of the system matrix $M_\tau(Z)$ is equal to

1. $(N - 1)p_f + m + 2n$, if $n < (N - 1)(m - p_f)$;
2. $n + Nm$, if $n \geq (N - 1)(m - p_f)$.

Proof. The proof is provided in Appendix at the end of this chapter. \square

Remark 3.2.7. When $p_f = m$, the system matrix $M_\tau(Z)$ has full-column normal rank; otherwise, n would be a negative integer which is impossible.

The main task of this chapter i.e. studying zeros of the blocked system, is resumed in the following. For this purpose, first some properties of the Kronecker canonical form of a matrix pencil are briefly reviewed. The system matrix of $\sum_\tau \forall \tau \in \{1, 2, \dots, N\}$ is actually a matrix pencil, and the Kronecker canonical form turns

out to be a very useful tool to obtain insight into zeros of (3.4) and the structure of the kernels associated with those zeros.

The main theorem on the Kronecker canonical form of a matrix pencil², is obtained from Van Dooren [1979].

Theorem 3.2.8. *Van Dooren [1979] Consider a matrix pencil $zR + S$. Then under the equivalence defined using pre- and postmultiplication by nonsingular constant matrices \tilde{P} and \tilde{Q} , there is a canonical quasidiagonal form*

$$\tilde{P}(zR + S)\tilde{Q} = \text{diag} \{L_{\epsilon_1}, \dots, L_{\epsilon_r}, \tilde{L}_{\eta_1}, \dots, \tilde{L}_{\eta_s}, zN - I, zI - K\}, \quad (3.7)$$

where

1. L_μ is the $\mu \times (\mu + 1)$ bidiagonal pencil

$$\begin{bmatrix} z & -1 & 0 & \dots & 0 & 0 \\ 0 & z & -1 & \dots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & z & -1 \end{bmatrix}. \quad (3.8)$$

2. \tilde{L}_μ is the $(\mu + 1) \times \mu$ transposed bidiagonal pencil

$$\begin{bmatrix} -1 & 0 & \dots & 0 & 0 \\ z & -1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & z & -1 \\ 0 & 0 & \dots & 0 & z \end{bmatrix}. \quad (3.9)$$

3. N is a nilpotent Jordan matrix.

4. K is in Jordan canonical form.

Furthermore, the possibility that $\mu = 0$ exists. The associated L_0 is deemed to have a column but not a row and \tilde{L}_0 is deemed to have a row but not a column, see Van Dooren [1979].

The following corollary can be directly derived easily from the above theorem and provides details about the vectors in the null space of the Kronecker canonical form. Because the matrices \tilde{P} and \tilde{Q} are nonsingular, it is trivial to translate these properties back to an arbitrary matrix pencil, including a system matrix.

Corollary 3.2.9. *With the same hypothesis as Theorem 3.2.8, and with $\Lambda(K)$ denoting the set of eigenvalues of K , the following hold*

²For more details about the canonical form of a matrix pencil, interested readers can refer to Grantmacher [1959], Van Dooren [1979], Van Dooren and Dewilde [1983] and the references listed therein.

1. For all $z \notin \Lambda(K)$, the kernel of the Kronecker canonical form has dimension equal to the number of matrices L_μ appearing in the form; likewise the co-kernel dimension is determined by the number of matrices \tilde{L}_μ .
2. The vector $[1 \ z \ z^2 \ \dots \ z^\mu]^T$ is the generator of the kernel of L_μ , a set of vectors

$$[0 \ \dots \ 0 \ 1 \ z \ z^2 \ \dots \ z^\mu \ 0 \ \dots \ 0]^T$$

are generators for the kernel of the whole canonical form which depend continuously on z , provided that $z \notin \Lambda(K)$; when $z \in \Lambda(K)$, the vectors form a subset of a set of generators.

3. When $z \in \Lambda(K)$ equals an eigenvalue of K , the dimension of the kernel jumps by the geometric multiplicity of that eigenvalue, the rank of the pencil drops below the normal rank by that geometric multiplicity, and there is an additional vector or vectors in the kernel apart from those defined in point 2, which are of the form $[0 \ 0 \ \dots \ v^T]^T$, where v is an eigenvector of K . Such a vector is orthogonal to all vectors in the kernel which are a linear combination of the generators listed in the previous point.
4. Let $\lambda_0 \in \Lambda(K)$ the associated kernel of the matrix pencil can be generated by two types of vectors: those which are the limit of the generators defined by adding extra zeros to vectors such as $[1 \ \lambda_0 \ \lambda_0^2 \ \dots \ \lambda_0^\mu]^T$ (these being the limits of the generators when $z \neq \lambda_0$ but continuously approaches λ_0); and those obtained by adjoining zeros to the eigenvector(s) corresponding to λ_0 , the latter set being orthogonal to the former set.

In the rest of this subsection, to study zeros of $M_\tau(Z)$, first a particular choice of $M_\tau(Z)$, namely $M_1(Z)$, is studied. Then later, the main result for zeros of $M_\tau(Z)$ $\forall \tau \in \{1, 2, \dots, N\}$ is introduced.

We begin by studying a square matrix generated from certain rows of $M_1(Z)$; these are the rows remaining after excluding certain output variables from consideration. To this end, we argue first that the first $n + Np_f$ rows of $M_1(Z)$ are linearly independent. For the submatrix formed by these rows, the system matrix of the blocked system is obtained by blocking the fast system defined by $\{A, B, C^f, D^f\}$, and accordingly has full-row normal rank, since the unblocked system is generic and square or fat under the condition $p_f \leq m$. Now define the square submatrix of $M_1(Z)$

$$N(Z) \triangleq \begin{bmatrix} ZI - A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix}, \quad (3.10)$$

such that $\text{normal rank}(N(Z)) = \text{normal rank}(M_1(Z))$, by including the first $n + Np_f$ rows of $M_1(Z)$ and followed by appropriate other rows of $M_1(Z)$ to meet the normal rank and squareness requirements. Note that there exists a permutation matrix P such that

$$PM_1(Z) = \begin{bmatrix} N(Z) \\ C_2 & D_2 \end{bmatrix}, \quad (3.11)$$

where \mathcal{C}_2 and \mathcal{D}_2 capture those rows of C_1 and D_1 that are not included in \mathcal{C}_1 and \mathcal{D}_1 , respectively.

Zeros of $N(Z)$ are studied in the following proposition. This is later used for obtaining results on zeros of $M_1(Z)$.

Proposition 3.2.10. *Let the matrix $N(Z)$ be the submatrix of $M_1(Z)$ formed via the procedure described. Then for generic values of the matrices A, B , etc. with $p_f \leq m$ and $Np_f + p_s > Nm$, for any finite Z_0 for which the matrix $N(Z_0)$ has less rank than its normal rank, its rank is one less than its normal rank.*

Proof. We distinguish two cases, $p_f = m$ and $p_f < m$. In the case $p_f = m$, $N(Z)$ is the system matrix for the system obtained by blocking the original system with slow outputs discarded. As such, the blocked system zeros are precisely the N -th powers of the unblocked system zeros Zamani et al. [2011]. For generic coefficient matrices, the unblocked system will have n distinct zeros; then the blocked system will have the same property. Further, the unblocked system will generically have a nonsingular direct feedthrough matrix, as will then the blocked system, so that \mathcal{D}_1 can be assumed to be nonsingular. It follows then that the zeros of the system with system matrix $N(Z)$ are identical with the eigenvalues of $A_1 - B_1 \mathcal{D}_1^{-1} C_1$, which are then distinct, and since this matrix is $n \times n$, the eigenvector associated with each zero will be uniquely defined to within a scaling constant. It follows easily that there is a unique vector (to within a scaling) in the kernel of $N(Z_0)$ where Z_0 is the zero of the blocked system.

In the case $p_f < m$. We study the co-kernel of $N(Z_0)$. Let Z_1, Z_2, \dots , be a sequence of complex numbers such that (a) $Z_i \rightarrow Z_0$ and (b) $\text{rank}(N(Z_i))$ equals the normal rank of $N(Z)$. From what has been described earlier using the Kronecker canonical form, we know that the sequence of co-kernels of $N(Z_i)$ converges, say to \mathcal{K} , with any vector in this limit also in the co-kernel of $N(Z_0)$. In addition, since $N(Z_0)$ has lower rank than the normal rank of $N(Z)$, the co-kernel, call it $\tilde{\mathcal{K}}$, will be strictly greater than \mathcal{K} . Suppose its dimension is at least two more than that of \mathcal{K} . We shall show this situation is nongeneric.

Select two vectors w_1, w_2 which are in $\tilde{\mathcal{K}}$ and which are orthogonal to \mathcal{K} . Then it is evident that there are two vectors call them v_1, v_2 , constructed from linear combinations of w_1, w_2 , which belong to $\tilde{\mathcal{K}}$, which are still orthogonal to \mathcal{K} , and which for some pair $r < s$ have 1 and 0 in the r -th entry and 0 and 1 in the s -th entry respectively. Choose v_1, v_2 so that firstly, s is maximal, and secondly, for that s then r is maximal. It is not difficult to see that this means that v_1 has zero entries beyond the r -th and v_2 has zero entries beyond the s -th.

Now again two cases must be considered. First, suppose that s obeys $n + Np_f + 1 \leq s \leq n + Nm$; in forming the product $v_2^T N(Z_0)$, the s -th entry of v_2 will be multiplying entries of $N(Z_0)$ defined using C^s, A, B, D^s . Consider an entry in the s -th row of $N(Z_0)$ and in the last m columns. Such an entry is an entry of D^s , and is independent of all other entries in $N(Z_0)$. Suppose this entry of D^s is continuously perturbed by a small amount. Then clearly v_1 remains in the co-kernel of $N(Z_0)$ but v_2 cannot.

The particular values of Z for which $N(Z)$ has rank less than its normal rank, i.e. the zeros of $N(Z)$, will depend continuously on the perturbation.

Accordingly, with a small enough perturbation, those not equal before perturbation to Z_0 will never change to Z_0 , and it is therefore guaranteed that with a small enough nonzero perturbation, the co-kernel of $N(Z_0)$ is reduced by one in dimension, though never to zero. If the original (before perturbation) co-kernel $\bar{\mathcal{K}}$ had dimension greater than two in excess of the dimension of \mathcal{K} , and the excess after perturbation is still greater than one, the argument can be repeated. Eventually, the co-kernel of $N(Z_0)$ will have an excess dimension over \mathcal{K} of 1, i.e. $N(Z_0)$ will have rank one less than the normal rank of $N(Z)$.

Now suppose that s obeys $s \leq n + Np_f$. Then the last $N(m - p_f)$ entries of each of v_1, v_2 are zero. Remove these entries to define two linearly independent vectors \tilde{v}_1, \tilde{v}_2 of length $n + Np_f$, which evidently satisfy

$$\tilde{v}_i^T \begin{bmatrix} ZI_n - A^N & -A^{N-1}B & \dots & -B \\ C^f & D^f & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-1} & C^f A^{N-2}B & \dots & D^f \end{bmatrix} = 0, \quad i = 1, 2. \quad (3.12)$$

The above equation contains a fat system matrix, corresponding to a blocked version of a fat time-invariant unblocked system. It can be concluded easily from the results provided in Zamani et al. [2011] that for generic values of the underlying matrices, there can be no Z_0 for which an equation such as (3.12) can even hold for a single nonzero \tilde{v}_i , let alone two linearly independent ones. This ends the proof. \square

The result of the previous proposition, although restricted to $\tau = 1$, enables us to establish the main result of this section, applicable for any τ . Before the main theorem of the section is provided, the following lemma from Chen et al. [2012] and Colaneri and Longhi [1995] needs to be recalled.

Lemma 3.2.11. *The pair (A, B) is reachable if and only if the pair $(A_\tau, B_\tau) \forall \tau \in \{1, 2, \dots, N\}$ is reachable.*

Theorem 3.2.12. *Consider the system $\Sigma_\tau, \forall \tau \in \{1, 2, \dots, N\}$, with $p_f \leq m$, and $Np_f + p_s > Nm$. Then for generic values of the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ the system matrix $M_\tau(Z) \forall \tau \in \{1, 2, \dots, N\}$, has rank equal to its normal rank for all finite nonzero values of Z_0 , and accordingly Σ_τ has no finite nonzero zero.*

Proof. We first focus on the case $\tau = 1$. Now, apart from the $p_s - N(m - p_f)$ rows of the C^s, D^s which do not enter the matrix $N(Z)$ defined by (3.10), choose generic values for the defining matrices, so that the conclusions of the preceding proposition are valid.

Let Z_a, Z_b, \dots be the finite set of Z for which $N(Z)$ has less rank than its normal rank (the set may have less than n elements, but never has more), and let w_a, w_b, \dots be vectors which are in the corresponding kernels (not co-kernels) and orthogonal to the subspace in the kernel obtained from the limit of the kernel of $N(Z)$ as $Z \rightarrow$

Z_a, Z_b, \dots etc. Now, due to the facts that $M_1(Z)$ and $N(Z)$ have the same normal rank and relation (3.11) holds, it follows that for generic Z , the kernels of $M_1(Z)$ and $N(Z)$ are identical (and may be both empty). Hence one can conclude that the subspace in the kernel obtained from the limit of the kernel of $N(Z)$ as Z approaches any of Z_a, Z_b, \dots etc. coincides with the subspace in the kernel obtained from the limit of the kernel of $M_1(Z)$ as $Z \rightarrow$ zeros of $M_1(Z)$.

Now, to obtain a contradiction, we suppose that the system matrix $M_1(Z)$ is such that, for $Z_0 \neq 0$, $M_1(Z_0)$ has rank less than its normal rank, i.e. the dimension of its kernel increases. Since the kernel of $M_1(Z_0)$ is a subspace of the kernel of $N(Z_0)$, Z_0 must coincide with one of the values of Z_a, Z_b, \dots and the rank of $M_1(Z_0)$ must be only one less than its normal rank; moreover, there must exist an associated nonzero w_1 unique up to a scalar multiplier, in the kernel of $M_1(Z_0)$ which is orthogonal to the limit of the kernel of $M_1(Z)$ as $Z \rightarrow Z_0$. Then w_1 is necessarily in the kernel of $N(Z_0)$, orthogonal to the limit of the kernel of $N(Z)$ as $Z \rightarrow Z_0$ and thus w_1 in fact must coincide to within a nonzero multiplier with one of the vectors w_a, w_b, \dots .

Write this w_1 as

$$w_1 = \begin{bmatrix} x_1 \\ u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}$$

and suppose the input sequence u_i is applied for $i = 1, 2, \dots, N$ to the original system, starting in initial state x_1 at time 1. Let y_1^f, y_2^f, \dots denote the corresponding fast outputs and y_N^s the slow output at time N . Break this up into two subvectors, y_N^{s1}, y_N^{s2} , where y_N^{s1} is associated with those rows of C^s, D^s which are included in $\mathcal{C}_1, \mathcal{D}_1$ (see (3.10)) and y_N^{s2} is related with the remaining rows of C^s and D^s . We have

$$N(Z_0)w_1 = \begin{bmatrix} Z_0 I_n - A^N & -A^{N-1}B & -A^{N-2}B & \dots & -B \\ C^f & D^f & 0 & \dots & 0 \\ C^f A & C^f B & D^f & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-1} & C^f A^{N-2}B & C^f A^{N-3}B & \dots & D^f \\ C^{s1} A^{N-1} & C^{s1} A^{N-2}B & C^{s1} A^{N-3}B & \dots & D^{s1} \end{bmatrix} w_1 = \begin{bmatrix} Z_0 x_1 - x_{N+1} \\ y_1^f \\ y_2^f \\ \vdots \\ y_N^f \\ y_N^{s1} \\ y_N^{s2} \end{bmatrix} = 0. \quad (3.13)$$

Now it must be true that $x_1 \neq 0$. For otherwise, we would have $N(Z)w_1 = 0$ for all Z , which would violate assumptions. Since $Z_0 \neq 0$, there must hold $x_{N+1} \neq 0$. Hence there cannot hold both $x_N = 0$ and $u_N = 0$. Consequently, we can always find C^{s2}, D^{s2} such that $y_N^{s2} = C^{s2}x_N + D^{s2}u_N \neq 0$, i.e. the slow output value is necessarily nonzero, no matter whether $w_1 = w_a, w_b$, etc. Equivalently, the equation $[\mathcal{C}_2 \ \mathcal{D}_2]w_1 = 0$ cannot hold. Hence, if $M_1(Z)$ defines a system with a finite zero and it is nonzero, this is a nongeneric situation. Hence, $M_1(Z)$ generically has rank equal to its normal rank for all finite nonzero Z . It now remains to show that this property carries over to

all $M_\tau(Z)$, $\tau \in \{2, 3, \dots, N\}$. First, note that the pair (A, B) is generically reachable; then by Lemma 3.2.11 the pair $(A_\tau, B_\tau) \forall \tau \in \{1, 2, \dots, N\}$, is also reachable. Consider $Z_\theta \in \mathbb{C} - \{0, \infty\}$; if Z_θ does not coincide with any eigenvalue of A_τ then

$$\text{rank}(M_\tau(Z_\theta)) = n + \text{rank}(V_\tau(Z_\theta)). \quad (3.14)$$

Hence, using the result of Proposition 3.4.3 (see Appendix), it is immediate that $\text{rank}(M_\tau(Z_\theta)) = \text{rank}(M_{\tau+1}(Z_\theta))$. If Z_θ does coincide with an eigenvalue of A_τ then $\text{rank}(V_\tau(Z_\theta))$ is ill-defined. However, since zeros of $M_\tau(Z)$, $\tau \in \{1, 2, \dots, N\}$, are invariant under state feedback and the pair (A_τ, B_τ) is reachable, one can easily find a state feedback to shift that eigenvalue Zhou et al. [1996] and then (3.14) is a well-defined equation and $\text{rank}(M_\tau(Z_\theta)) = \text{rank}(M_{\tau+1}(Z_\theta))$. Thus, we can conclude that all $M_\tau(Z)$, $\tau \in \{1, 2, \dots, N\}$, generically have no finite nonzero zeros. This ends the proof. \square

3.3 Blocked Systems with Generic Parameters - Zeros at the Origin and Infinity

In the previous section, zeros of tall blocked systems with generic parameters for the choice of finite nonzero zeros were studied. In this section, zeros of these systems are investigated for choices of zeros at zero and infinity. As in the previous section, it is convenient to break up our examination of tall systems into two separate cases based on the relation between p_f and m .

The following result, perhaps surprisingly, relates zeros of the system Σ_τ at infinity to zeros of the system $\Sigma_{N-\tau+1}$ at the origin and conversely.

Lemma 3.3.1. *Consider the family of systems $\Sigma_\tau \forall \tau \in \{1, 2, \dots, N\}$, where the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ assume generic values. Then the following fact holds. Σ_τ has κ zeros at $Z = 0$ and α zeros at $Z = \infty$ if and only if $\Sigma_{N-\tau+1}$ has α zeros at $Z = 0$ and κ zeros at $Z = \infty$.*

Proof. Consider a reverse-time description of the system (3.2), namely

$$\begin{aligned} x_{t-1} &= A^{-1}x_t - A^{-1}Bu_{t-1}, & t &= 1, 2, \dots, \\ y_{t-1}^f &= C^f x_{t-1} + D^f u_{t-1} & t &= 1, 2, \dots, \\ &= C^f A^{-1}x_t + (D^f - C^f A^{-1}B)u_{t-1}, & & \\ y_{t-1}^s &= C^s x_{t-1} + D^s u_{t-1} & t &= 1, N+1, \dots \\ &= C^s A^{-1}x_t + (D^s - C^s A^{-1}B)u_{t-1}, & & \end{aligned} \quad (3.15)$$

and define the following matrices

$$\begin{aligned} \tilde{A} &\triangleq A^{-1}, & \tilde{B} &\triangleq -A^{-1}B, \\ \tilde{C}^f &\triangleq C^f A^{-1}, & \tilde{D}^f &\triangleq D^f - C^f A^{-1}B, \\ \tilde{C}^s &\triangleq C^s A^{-1}, & \tilde{D}^s &\triangleq D^s - C^s A^{-1}B, \end{aligned} \quad (3.16)$$

which are still in a generic setting since the genericity of $\{A, B, C^f, D^f, C^s, D^s\}$ is

assumed. Note that the matrix A^{-1} is well-defined, since A is generically full rank. Recall the blocking procedure introduced in (3.3) for a given value of τ ; we can obtain the blocked time-invariant system associated with the system (3.15) again a reverse-time system as

$$\begin{aligned} x_{t-N}^\tau &= \tilde{A}_\tau x_t^\tau + \tilde{B}_\tau U_{t-N}^\tau, \\ Y_{t-N}^\tau &= \tilde{C}_\tau x_t^\tau + \tilde{D}_\tau U_{t-N}^\tau, \end{aligned} \quad (3.17)$$

where $t = N, 2N, \dots$, and

$$\begin{aligned} \tilde{A}_\tau &\triangleq \tilde{A}^N, \\ \tilde{B}_\tau &\triangleq [\tilde{B} \quad \tilde{A}\tilde{B} \quad \dots \quad \tilde{A}^{N-2}\tilde{B} \quad \tilde{A}^{N-1}\tilde{B}], \\ \tilde{C}_\tau &\triangleq [\tilde{A}^{(N-1)T}\tilde{C}^T \quad \dots \quad \tilde{C}^T \quad \tilde{A}^{(\tau-1)T}\tilde{C}^T]^T, \\ \tilde{D}_\tau &\triangleq \left[\begin{array}{ccc|c} \tilde{D}^f & \dots & \tilde{C}^f \tilde{A}^{N-3}\tilde{B} & \tilde{C}^f \tilde{A}^{N-2}\tilde{B} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \tilde{D}^f & \tilde{C}^f \tilde{B} \\ 0 & \dots & 0 & \tilde{D}^f \\ \hline & & \tilde{D}_\tau^s & \end{array} \right]. \end{aligned} \quad (3.18)$$

In the latter expression, when $\tau > 1$ the matrix \tilde{D}_τ^s is equal to $[0 \dots 0 \tilde{D}^s \dots \tilde{C}^s \tilde{A}^{\tau-2}\tilde{B}]$, with $N - \tau$ zero blocks of size $p_s \times m$, while, when $\tau = 1$, it becomes $[0 \dots 0 \tilde{D}^s]$. Now let us introduce the N -step backward operator ζ , such that $\zeta x_t = x_{t-N}$. Then the transfer function $\tilde{V}_\tau(\zeta) \triangleq \tilde{C}_\tau(\zeta I - \tilde{A}_\tau)^{-1}\tilde{B}_\tau + \tilde{D}_\tau$ is associated with the blocked system (3.17). It can be easily checked through simple computations that this transfer function is connected to the transfer function $V_\tau(Z)$ associated with the system Σ_τ at the points zero and infinity through the equalities

$$\tilde{V}_\tau(0) = \lim_{Z \rightarrow \infty} V_\tau(Z), \quad \lim_{\zeta \rightarrow \infty} \tilde{V}_\tau(\zeta) = V_\tau(0). \quad (3.19)$$

We now comment on another alternative approach to obtain the above equalities. Note that when one deals with unblocked linear time-invariant systems the above equalities are well-known. Recall that Theorem 2.3.9 and Theorem 2.3.11 relate the zeros of the blocked systems to those of their associated unblocked systems when $z = \infty$ and $z = 0$, respectively. Hence, one can readily conclude that the above equalities hold.

Define the system matrix associated with the system (3.17) as

$$\tilde{M}_\tau(\zeta) \triangleq \begin{bmatrix} \zeta I - \tilde{A}_\tau & -\tilde{B}_\tau \\ \tilde{C}_\tau & \tilde{D}_\tau \end{bmatrix}. \quad (3.20)$$

For our purpose in this chapter, we define the following equalities.

$$\begin{aligned}\text{rank} \left(\lim_{Z \rightarrow \infty} M_\tau(Z) \right) &\triangleq n + \text{rank} (D_\tau), \\ \text{rank} \left(\lim_{\zeta \rightarrow \infty} \tilde{M}_\tau(\zeta) \right) &\triangleq n + \text{rank} (\tilde{D}_\tau).\end{aligned}\tag{3.21}$$

Then using the equation (3.19) one can write

$$\begin{aligned}\text{rank} \left(\lim_{Z \rightarrow \infty} M_\tau(Z) \right) &= \text{rank} (\tilde{M}_\tau(0)), \\ \text{rank} \left(\lim_{\zeta \rightarrow \infty} \tilde{M}_\tau(\zeta) \right) &= \text{rank} (M_\tau(0)).\end{aligned}\tag{3.22}$$

Again, note that the above equalities are well-defined since, due to the genericity assumption of the matrix A , the matrices A_τ and \tilde{A}_τ do not have any eigenvalues at the origin. Now, by comparing (3.5) and (3.18), one can verify that there exist permutation matrices Q_1 and Q_2 such that $Q_1 M_\tau(\zeta) Q_2 = \Psi_\tau(\zeta)$, and $\Psi_\tau(\zeta)$ is exactly $M_{N-\tau+1}(Z)$ when $\tilde{A}, \tilde{B}, \tilde{C}^f, \tilde{C}^s, \tilde{D}^f, \tilde{D}^s, \zeta$ are replaced by $A, B, C^f, D^f, C^s, D^s, Z$, accordingly. Since the parameter matrices A, B, C, D assume generic values, we have the following equalities

$$\begin{aligned}\text{rank} \left(\lim_{\zeta \rightarrow \infty} \tilde{M}_\tau(\zeta) \right) &= \text{rank} \left(\lim_{Z \rightarrow \infty} M_{N-\tau+1}(Z) \right), \\ \text{rank} (\tilde{M}_\tau(0)) &= \text{rank} (M_{N-\tau+1}(0)).\end{aligned}\tag{3.23}$$

Then, by combining equations (3.22) and (3.23) we obtain

$$\text{rank} \left(\lim_{Z \rightarrow \infty} M_\tau(Z) \right) = \text{rank} (M_{N-\tau+1}(0))\tag{3.24}$$

and

$$\text{rank} (M_\tau(0)) = \text{rank} \left(\lim_{Z \rightarrow \infty} M_{N-\tau+1}(Z) \right).\tag{3.25}$$

Thus, by using equations (3.24), (3.25) and the fact that the normal rank of $M_\tau(Z)$ does not depend on τ (see Proposition 3.4.3 in Appendix), the conclusion of the lemma readily follows. \square

3.3.1 Case $p_f > m$

Theorem 3.3.2. *For a generic choice of the matrices $\{A, B, C^s, C^f, D^s, D^f\}$, $p_f > m$, the system matrix of $\Sigma_\tau \forall \tau \in \{1, 2, \dots, N\}$, has full-column rank at $Z = 0$ and $Z = \infty$, and accordingly Σ_τ has no zero at $Z = 0$ and $Z = \infty$.*

Proof. We first consider the zeros at $Z = 0$. It was shown in Zamani et al. [2011] that $M^f(0)$, where the system matrix $M^f(0)$ can be formed by deleting rows of $M_1(0)$ which are related to C^s and D^s , has full-column rank at $Z = 0$ for generic parameter matrices A, B , etc. Then it is immediate that $M_\tau(0) \forall \tau \in \{1, 2, \dots, N\}$ has full-column rank, implying that the system Σ_τ has no zero at $Z = 0$. Next, consider zeros at

infinity. Using Lemma 3.3.1, it follows that $M_\tau(Z) \forall \tau \in \{1, \dots, N\}$ is full-column rank. Hence, Σ_τ has no zeros at infinity. \square

3.3.2 Case $p_f \leq m, Np_f + p_s > Nm$

In this subsection, zeros of tall blocked systems at infinity and the origin are studied. First, zeros at infinity are examined. According to Definition 2.3.2, the rank of matrix D_τ plays a crucial role in the determination of the zeros at infinity. At this stage, the result of Proposition 3.2.5 can be exploited for determining the multiplicity of zeros at infinity.

Theorem 3.3.3. *Consider the system $\Sigma_\tau \forall \tau \in \{1, 2, \dots, N\}$, with $p_f \leq m$ and $Np_f + p_s > Nm$. Assume that the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ take generic values. Then $M_\tau(Z)$ has zeros at $Z = \infty$ with multiplicity equal to*

1. 0 if $n \leq (N - \tau)(m - p_f)$;
2. $n - (N - \tau)(m - p_f)$ if $(N - \tau)(m - p_f) < n \leq (N - 1)(m - p_f)$;
3. $(\tau - 1)(m - p_f)$ if $n > (N - 1)(m - p_f)$.

Proof. Denote by σ the multiplicity of zeros at infinity. Then, by Definition 3.2.1, $\sigma = \text{normal rank}(M_\tau(Z)) - n - \text{rank}(D_\tau)$. Consider the following cases.

1. $n \leq (N - \tau)(m - p_f)$. From Theorem 3.2.6 $\text{normal rank}(M_\tau(Z)) = (N - 1)p_f + m + 2n$, while Proposition 3.2.5 yields that $\text{rank}(D_\tau) = (N - 1)p_f + m + n$. Then one can easily conclude that $\sigma = 0$.
2. $(N - \tau)(m - p_f) < n \leq (N - 1)(m - p_f)$. From Theorem 3.2.6, still the equality $\text{normal rank}(M_\tau(Z)) = (N - 1)p_f + m + 2n$ holds, while now Proposition 3.2.5 yields $\text{rank}(D_\tau) = (\tau - 1)p_f + (N - \tau + 1)m$. Hence, in this case $\sigma = n - (N - \tau)(m - p_f)$.
3. $n > (N - 1)(m - p_f)$. In this case, from Theorem 3.2.6 one can observe that the system matrix has full-column normal rank, namely $n + Nm$, while, according to Proposition 3.2.5, the rank of D_τ is still $(\tau - 1)p_f + (N - \tau - 1)m$. Then one can determine that $\sigma = (\tau - 1)(m - p_f)$.

\square

The following corollary studies zeros at the origin.

Corollary 3.3.4. *Consider the system $\Sigma_\tau \forall \tau \in \{1, \dots, N\}$, with $p_f \leq m$ and $Np_f + p_s > Nm$. Assume that the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ take generic values. Then $M_\tau(Z)$ has zeros at $Z = 0$ with multiplicity equal to*

1. 0 if $n \leq (\tau - 1)(m - p_f)$;
2. $n - (\tau - 1)(m - p_f)$ if $(\tau - 1)(m - p_f) < n \leq (N - 1)(m - p_f)$;

3. $(N - \tau)(m - p_f)$ if $n > (N - 1)(m - p_f)$.

Proof. Pick $\bar{\tau}$ in the set $\{1, 2, \dots, N\}$ and consider the following situations.

1. $n \leq (N - \bar{\tau})(m - p_f)$. In this case, from Theorem 3.3.3 one can see that the system $\Sigma_{\bar{\tau}}$ has no zeros at infinity. Then, recalling Lemma 3.3.1, we also have that $\Sigma_{N-\bar{\tau}+1}$ has no zeros at $Z = 0$. Then, by defining $\tau = N - \bar{\tau} + 1$ and substituting in the inequality $n \leq (N - \bar{\tau})(m - p_f)$, one can easily obtain that, when $n \leq (\tau - 1)(m - p_f)$, the system $\Sigma_{N-\bar{\tau}+1} \equiv \Sigma_{\tau}$ has no zeros at $Z = 0$.
2. $(N - \bar{\tau})(m - p_f) < n \leq (N - 1)(m - p_f)$. In this case, $\Sigma_{\bar{\tau}}$ has $n - (N - \bar{\tau})(m - p_f)$ zeros at infinity. Using the same arguments employed for the previous case, one can reason that, when $(\tau - 1)(m - p_f) < n \leq (N - 1)(m - p_f)$, Σ_{τ} has $n - (\tau - 1)(m - p_f)$ zeros at $Z = 0$.
3. $n > (N - 1)(m - p_f)$. Again, since $\Sigma_{\bar{\tau}}$ has $(\bar{\tau} - 1)(m - p_f)$ zeros at infinity, one obtains that Σ_{τ} has $(N - \tau)(m - p_f)$ zeros at the origin.

□

We now provide the following numerical example which supports our results in this chapter.

Example 3.3.5. Consider the system Σ_{τ} and suppose that $m = 4$, $p_f = 3$, $p_s = 9$, $n = 8$ and the block size i.e. N is equal to 6. Then one can verify that for a generic set of matrices A, B, C^f, D^f, C^s and D^s the system Σ_{τ} has no finite nonzero zeros. It turns out that the corresponding system matrix has full-column normal rank (note that in this setup $n > (N - 1)(m - p_f)$). Furthermore, the following table summarize zeros at the origin and infinity for different values of τ .

Table 3.1: Zeros at the origin and infinity for the system in Example 3.3.5

Value of τ \ number of zeros	at the origin	at infinity
1	5	0
2	4	1
3	3	2
4	2	3
5	1	4
6	0	5

Remark 3.3.6. The above results reveal that, assuming A, B , etc. are generic with $p_f \leq m$ and $Np_f + p_s > Nm$, when $\tau = 1$ all zeros are at the origin and no zero at infinity. Conversely, when $\tau = N$ all zeros are at infinity and there are no zeros at the origin. Furthermore, when $\tau = 1$ there is always at least one zero at the origin, while when $\tau = N$ there is always at least one zero at infinity (unless one considers a system with no dynamics, i.e. a system with $n = 0$).

Remark 3.3.7. When $p_f = m$, the conditions given in Theorem 3.3.3 and the subsequent Corollary on the presence of zeros at $Z = 0$ and $Z = \infty$ shrink to empty sets. Then, it follows that Σ_τ has neither zeros at the origin nor at infinity.

Remark 3.3.8. In some special cases depending on the state, input and output dimensions, Σ_τ may have zeros at the origin or at infinity for some values of τ but be completely zero-free for other values of τ . For example, consider Σ_τ for particular choice of $n = 5$, $m = 5$, $p_f = 3$, $p_s = 24$ and $N = 8$ which has zeros for all values of τ , except for $\tau = 4, 5$. In these particular cases, the system Σ_τ is totally zero-free. This can be easily checked by using Theorem 3.3.3 and the subsequent Corollary.

Table 3.2: Summarizing the results obtained in this chapter.

Zero \ Region	$p_f \geq m$	$p_f < m,$ $Np_f + p_s > Nm$
Finite nonzero zeros	No	No
Zeros at zero	No	Zeros can be at these points depending on τ .
Zeros at infinity	No	

Various theorems have been introduced in this chapter regarding zeros of the system Σ_τ given a generic underlying multirate system. Accordingly, the results obtained in this chapter are summarized in Table 3.2.

3.4 Summary

Zeros of tall blocked systems were studied in this chapter. Zeros were specifically explored when the defining parameters of the associated underlying multirate system obtain generic values. To study zeros, we needed to investigate the generic rank of the system matrix; hence, as a part of the investigation, the generic rank assumed by the system matrix of the blocked system and the transfer function of that system were completely calculated. It was demonstrated that tall blocked systems have no finite nonzero zeros. However, depending on the relevant integer parameters i.e. input, state, and output dimensions and ratio of sampling rates, blocked systems may have zeros at $Z = 0$ or $Z = \infty$ or both, or even be completely zero-free.

Appendix

Proof of Proposition 3.2.5

We first need to introduce the following lemma.

Lemma 3.4.1. Consider a generic pair of matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Then, given $v \in \mathbb{N}$, the matrix

$$C = [B \quad AB \quad \dots \quad A^{v-1}B] \quad (3.26)$$

is always full rank, i.e. its rank is equal to

1. its number of rows, n , if $n \leq vm$;
2. its number of columns, vm , if $n > vm$.

Proof. Since the case $m \geq n$ is straightforward, we focus on the case $n > m$. The statement can be proven by finding a pair (A, B) such that the matrix C attains full rank, since it means that this happens for any generic pair of such matrices. Accordingly, choose the following matrices

$$A = \begin{bmatrix} 0_{m \times (n-m)} & I_m \\ I_{n-m} & 0_{(n-m) \times m} \end{bmatrix}, \quad B = \begin{bmatrix} I_m \\ 0_{(n-m) \times m} \end{bmatrix}, \quad (3.27)$$

regarding which we point out the following properties.

1. The matrix A acts as a circular left-shift operator matrix through m positions and can be written in terms of the canonical basis of \mathbb{R}^n , as $A = [e_{m+1} \dots e_n \ e_1 \dots e_m]$. Then, if for example $n > 3m + 1$, one has $A^2 = [e_{2m+1} \dots e_n \ e_1 \dots e_{2m}]$, $A^3 = [e_{3m+1} \dots e_n \ e_1 \dots e_{3m}]$.
2. The matrix B selects the first m columns of any matrix which premultiplies it. Furthermore, the columns of B correspond to e_1, \dots, e_m .

Based on these considerations, we then have

$$B = [e_1 \dots e_m], \quad AB = [e_{m+1} \dots e_{2m}], \\ A^2B = [e_{2m+1} \dots e_{3m}] \dots A^{v-1}B = [e_{(v-1)m+1} \dots e_{vm}],$$

where for simplicity we have adopted the notation $e_{(kn+i)} = e_i$, $i = 1, \dots, n$, $k \in \mathbb{N}$. Then, it is easy to conclude that

1. if $n \leq vm$, all the vectors of the canonical basis of \mathbb{R}^n enter in the matrix C at least once, and thus C is full-row rank;
2. if $n > vm$, there are vm distinct vectors of the canonical basis of \mathbb{R}^n entering in the matrix C and thus C is full-column rank.

□

We can now prove Proposition 3.2.5. For the sake of brevity, we treat only the case $n \geq m$, since the case $n < m$ is virtually the same. Fix τ and first assume

$n \leq (N - \tau)(m - p_f)$. We consider a particular system, defined by the matrices

$$\begin{aligned} A &= \begin{bmatrix} 0_{(m-p_f) \times (n-m+p_f)} & I_{m-p_f} \\ I_{n-m+p_f} & 0_{(n-m+p_f) \times (m-p_f)} \end{bmatrix}, & B &= \begin{bmatrix} I_{m-p_f} & 0_{(m-p_f) \times p_f} \\ 0_{(n-m+p_f) \times m} \end{bmatrix}, \\ C^f &= 0_{p_f \times n}, & C^s &= \begin{bmatrix} I_n \\ 0_{(p_s-n) \times n} \end{bmatrix}, \\ D^f &= \begin{bmatrix} 0_{p_f \times (m-p_f)} & I_{p_f} \end{bmatrix}, & D^s &= \begin{bmatrix} 0_{n \times m} \\ I_{m-p_f} & 0_{(m-p_f) \times p_f} \\ 0_{(p_s+p_f-n-m) \times m} \end{bmatrix}. \end{aligned}$$

Note that, under the working assumptions, the dimensions of the various matrices involved in the construction of such system are consistent. In particular, since $n \leq (N - \tau)(m - p_f)$ and, by assumption of tallness, $p_s > N(m - p_f)$, one has

$$n + m \leq (N - \tau)(m - p_f) + m \leq (N - 1)(m - p_f) + m = p_f + N(m - p_f) < p_f + p_s$$

and so $p_s + p_f - n - m > 0$. Below, we adopt the notation that, if a submatrix has zero rows or columns, then it does not appear in the relative matrix. Before writing D_τ explicitly, we focus on the submatrix

$$[C^s A^{N-\tau-1} B \quad C^s A^{N-\tau-2} B \quad \dots \quad C^s B],$$

which enters in the block row associated with the slow dynamics of the blocked system. Due to the structure of C^s , a first rewriting yields

$$[A^{N-\tau-1} B \quad A^{N-\tau-2} B \quad \dots \quad B]. \quad (3.28)$$

Now, we point out the following properties of A and B .

1. The matrix A acts as a circular left-shift operator matrix through $m - p_f$ positions. Furthermore, the columns of A are orthogonal.
2. The matrix B selects the first $m - p_f$ columns of any matrix, which premultiplies it. The other p_f columns of the resulting matrix are set to zero. Furthermore, the nonzero columns of B correspond to e_1, \dots, e_{m-p_f} .

Based on these considerations, we then have

$$\begin{aligned} B &= \begin{bmatrix} e_1 & \dots & e_{m-p_f} & 0_{n \times p_f} \end{bmatrix}, \\ AB &= \begin{bmatrix} e_{m-p_f+1} & \dots & e_{2(m-p_f)} & 0_{n \times p_f} \end{bmatrix}, \\ A^2 B &= \begin{bmatrix} e_{2(m-p_f)+1} & \dots & e_{3(m-p_f)} & 0_{n \times p_f} \end{bmatrix}, \\ &\vdots \\ A^{N-\tau-1} B &= \begin{bmatrix} e_{(N-\tau-1)(m-p_f)+1} & \dots & e_{(N-\tau)(m-p_f)} & 0_{n \times p_f} \end{bmatrix}, \end{aligned}$$

We define

$$\Delta_a \triangleq \begin{bmatrix} D^f & 0 & \dots & 0 \\ C^f B & D^f & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-\tau-2} B & \dots & D^f & 0 \end{bmatrix}, \quad \Delta_b \triangleq \begin{bmatrix} C^f A^{N-\tau-1} B & \dots & C^f B & D^f \\ C^s A^{N-\tau-1} B & \dots & C^s B & D^s \end{bmatrix},$$

so that

$$\Delta_1 = \begin{bmatrix} \Delta_a \\ \Delta_b \end{bmatrix}$$

and $\text{rank}(\Delta_1) \leq \text{rank}(\Delta_a) + \text{rank}(\Delta_b)$. Note that, Δ_1 is a tall matrix, since it includes the slow rate outputs whose dimension ensure tallness in the whole system. Hence, its maximum achievable rank is given by the number of its columns, namely $(N - \tau + 1)m$. Thus we can find a first upper bound for the rank of D_τ , that is

$$\text{rank}(D_\tau) \leq (N - \tau + 1)m + (\tau - 1)p_f, \quad (3.31)$$

and this will be used below. Meanwhile, we focus on the analysis of Δ_a . It is well-known (see Zamani et al. [2011]) that, due to genericity of the matrix D^f , Δ_a is full-row rank, namely $(N - \tau)p_f$. For Δ_b , we consider the following factorization

$$\Delta_b = \begin{bmatrix} C^f & D^f \\ C^s & D^s \end{bmatrix} \begin{bmatrix} A^{N-\tau-1} B & A^{N-\tau-2} B & \dots & B & 0 \\ 0 & 0 & \dots & 0 & I_m \end{bmatrix} \triangleq H\mathcal{R}. \quad (3.32)$$

Since by assumption $n \leq (N - \tau)(m - p_f)$, from Lemma 3.4.1 one can see that the matrix \mathcal{R} is full-row rank, namely $n + m$. Thus, the rank of Δ_b is determined by $H \in \mathbb{R}^{(p_f + p_s) \times (n + m)}$. On the one hand, assumption of tallness of the blocked system ensures $p_s > N(m - p_f)$; on the other hand, since $n \leq (N - \tau)(m - p_f)$, one has $n + m < p_s + p_f$. Hence Δ_b is tall, and so generically $\text{rank}(\Delta_b) = n + m$ and $\text{rank}(\Delta_1) \leq \text{rank}(\Delta_a) + \text{rank}(\Delta_b) = (N - \tau)p_f + n + m$, which in turn implies

$$\text{rank}(D_\tau) = \text{rank}(\Delta_1) + \text{rank}(\Delta_2) \leq (N - \tau)p_f + n + m + (\tau - 1)p_f = (N - 1)p_f + n + m,$$

which corresponds to the lower bound found previously.

In order to complete our proof, it remains to analyze the case $n > (N - \tau)(m - p_f)$. To do so, we first make an observation concerning the case $n = (N - \tau)(m - p_f)$, which was covered in the first part of the proof. In this particular case, $\text{rank}(D_\tau) = (N - 1)p_f + n + m = (N - \tau + 1)m + (\tau - 1)p_f$, which corresponds to the upper bound on rank of D_τ given by (3.31). Now, the proof for the case $n > (N - \tau)(m - p_f)$ can be completed by showing that such an upper bound is attained by any generic tall system with $n = (N - \tau)(m - p_f) + q$, $q \in \mathbb{N}$. This can be verified by choosing

the system

$$A = \begin{bmatrix} 0_{(m-p_f) \times (N-\tau-1)(m-p_f)} & I_{m-p_f} & 0_{(m-p_f) \times q} \\ I_{(N-\tau-1)(m-p_f)} & 0_{(N-\tau-1)(m-p_f) \times (m-p_f)} & 0_{((N-\tau-1)(m-p_f) \times q)} \\ 0_{q \times ((N-\tau-1)(m-p_f))} & 0_{q \times (m-p_f)} & 0_{q \times q} \end{bmatrix},$$

$$B = \begin{bmatrix} I_{m-p_f} & 0_{(m-p_f) \times p_f} \\ 0_{(n-m+p_f) \times m} \end{bmatrix}, \quad (3.33)$$

$$C^f = 0_{p_f \times n}, \quad D^f = \begin{bmatrix} 0_{p_f \times (m-p_f)} & I_{p_f} \end{bmatrix},$$

$$C^s = \begin{bmatrix} I_{(N-\tau)(m-p_f)} & 0_{(N-\tau)(m-p_f) \times q} \\ 0_{(p_s-(N-\tau)(m-p_f)) \times n} \end{bmatrix}, \quad D^s = \begin{bmatrix} 0_{(N-\tau)(m-p_f) \times m} \\ I_{m-p_f} & 0_{(m-p_f) \times p_f} \\ 0_{(p_s+p_f-m-(N-\tau)(m-p_f)) \times m} \end{bmatrix},$$

which generates a matrix D_τ equal to the one generated by the system (3.28), when $n = (N - \tau)(m - p_f)$. Since we have previously proven that, in that case, the rank is $(\tau - 1)p_f + (N - \tau + 1)m$ (which is also the maximum rank achievable), then also for any $n > (N - \tau)(m - p_f)$ we have $\text{rank}(D_\tau) = (\tau - 1)p_f + (N - \tau + 1)m$. This completes the proof.

Proof of Theorem 3.2.6

Before proving our result on the normal rank of $M_\tau(Z)$, we need to introduce three preliminary results. The following lemma is adopted from Bittanti and Colaneri [2009] and modified for our own purpose.

Lemma 3.4.2. *The transfer function $V_\tau(Z)$ associated with the blocked system (3.4) has the following property*

$$V_{\tau+1}(Z) = \begin{bmatrix} 0 & I_{p_1(N-1)} & 0 \\ ZI_{p_f} & 0 & 0 \\ 0 & 0 & I_{p_s} \end{bmatrix} V_\tau(Z) \begin{bmatrix} 0 & Z^{-1}I_m \\ I_{m(N-1)} & 0 \end{bmatrix}, \quad (3.34)$$

where $\tau \in \{1, 2, \dots, N-1\}$.

Proposition 3.4.3. *The normal rank of the system matrix $M_\tau(Z)$ is the same for every value of $\tau \in \{1, 2, \dots, N\}$.*

Proof. Using the above lemma, one can easily conclude that the transfer function matrices $V_{\tau+1}(Z_0)$ and $V_\tau(Z_0)$ have the same rank provided that Z_0 does not belong to the finite set of poles of the $V_\tau(Z)$ (which is the same as that of $V_{\tau+1}(Z)$) and $Z_0 \notin \{0, \infty\}$. Hence, one can determine that $V_{\tau+1}(Z)$ and $V_\tau(Z)$ have the same normal rank and so do their associated system matrices i.e. $M_{\tau+1}(Z)$ and $M_\tau(Z)$. \square

Proposition 3.4.4. Consider the system Σ_1 (i.e. the blocked system obtained with $\tau = 1$), with $p_f < m$, $Np_f + p_s > Nm$ and generic values of the defining matrices $\{A, B, C^f, C^s, D^f, D^s\}$. Then

1. if $n \leq (N - 1)(m - p_f)$, the matrix D_1 has rank equal to $(N - 1)p_f + m + n$;
2. iff $n > (N - 1)(m - p_f)$, the matrix D_1 has full-column rank, namely Nm .

Proof. The proof follows easily from Proposition 3.2.5, by letting $\tau = 1$. \square

We are now ready to prove Theorem 3.2.6. Here, we focus on the matrix $M_1(Z)$; every result on its normal rank can be easily extended to any value of $\tau = \{2, \dots, N\}$ using Proposition 3.4.3.

Consider the matrix D_1 and define $r \triangleq \text{rank}(D_1)$; note that the condition of tallness of the system implies $r \leq Nm$. Define the full-row rank matrix $\bar{D}_1 \in \mathbb{R}^{r \times Nm}$, obtained by discarding a proper number of linearly dependent rows of D_1 . Similarly, define \bar{C}_1 discarding the corresponding rows from C_1 . Without loss of generality assume A diagonal. This hypothesis is not limiting; in fact, under a generic setting, A has n distinct eigenvalues and so it is diagonalizable. If one considers a change of basis T such that $T^{-1}AT$ is diagonal, then the other parameter matrices $T^{-1}B$ and CT are still in a generic setting. Define $\bar{M}_1(Z)$ as follows.

$$\bar{M}_1(Z) = \begin{bmatrix} Z - a_1^N & \dots & 0 & -b_1^T \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & Z - a_n^N & -b_n^T \\ \bar{c}_{1,1} & \dots & \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix}, \quad (3.35)$$

where the a_i 's represent the diagonal elements of A , b_i^T is the i -th row of B_1 and $\bar{c}_{i,1}$ is the i -th column of \bar{C}_1 . Consider the submatrix $[\bar{c}_{1,n} \ \bar{D}_1]$. Since \bar{D}_1 is full-row rank, also this matrix is full-row rank. Consider the equation

$$v^T [\bar{c}_{1,n} \ \bar{D}_1] = [Z - a_n^N \ -b_n^T], \quad (3.36)$$

in which v and Z are yet to be specified and which can be rewritten as

$$\begin{cases} v^T \bar{c}_{1,n} &= Z - a_n^N \\ v^T \bar{D}_1 &= -b_n^T \end{cases}. \quad (3.37)$$

Since \bar{D}_1 is full-row rank there exists at most one vector \bar{v}^T satisfying the second relation. Clearly, if one were to insert such a vector in the first relation, there could exist only one value $Z_n \in \mathbb{C}$ such that this equation is satisfied. Choose $Z \neq Z_n$ and consider the submatrix

$$\begin{bmatrix} 0 & Z - a_n^N & -b_n^T \\ \bar{c}_{1,n-1} & \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix}, \quad (3.38)$$

which is clearly full-row rank, namely $r + 1$. Write the equation

$$v^T \begin{bmatrix} 0 & Z - a_n^N & -b_n^T \\ \bar{c}_{1,n-1} & \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix} = [Z - a_{n-1}^N \quad 0 \quad -b_{n-1}^T], \quad (3.39)$$

which in turn can be rewritten as

$$\begin{cases} v^T \begin{bmatrix} 0 \\ \bar{c}_{1,n-1} \end{bmatrix} = Z - a_{n-1}^N \\ v^T \begin{bmatrix} Z - a_n^N & -b_n^T \\ \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix} = [0 \quad -b_{n-1}^T] \end{cases}. \quad (3.40)$$

Again, the second relation admits at most one solution, which is compatible with the first equation for only one value $Z_{n-1} \in \mathbb{C}$. Hence, choosing $Z \notin \{Z_n, Z_{n-1}\}$ one can build the matrix

$$\begin{bmatrix} 0 & Z - a_{n-1}^N & 0 & -b_{n-1}^T \\ 0 & 0 & Z - a_n^N & -b_n^T \\ \bar{c}_{1,n-2} & \bar{c}_{1,n-1} & \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix}, \quad (3.41)$$

which is full-row rank, namely $r + 2$, and repeat the previous steps until all the rows containing the a_i^N 's and the b_i^T 's, $i \in \{1, \dots, n\}$, are considered. This procedure ends after n iterations, when all the rows of the matrix $\bar{M}_1(Z)$ are included; clearly the rank turns out to be $r + n$. Since $\bar{M}_1(Z)$ is a submatrix of $M_1(Z)$, the normal rank of $M_1(Z)$ is greater than or equal to $r + n$. There are two cases in the theorem statement. Treating the second one first, suppose $n \geq (N - 1)(m - p_f)$. Recalling Proposition 3.4.4, $r = Nm$; hence normal rank $(\bar{M}_1(Z)) = n + Nm$ and $M_1(Z)$ is full normal rank.

For the second case, suppose $n < (N - 1)(m - p_f)$. In this case, from Proposition 3.4.4 we have $r = (N - 1)p_f + m + n$, hence normal rank $(M_1(Z)) \geq$ normal rank $(\bar{M}_1(Z)) = (N - 1)p_f + m + 2n$. Now, consider the submatrix formed by the first $n + (N - 1)p_f$ rows of $M_1(Z)$. Such a submatrix is full normal rank, since it can be seen also as a submatrix of the system matrix

$$\begin{bmatrix} ZI_n - A^N & -A^{N-1}B & \dots & -B \\ C^f & D^f & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-1} & C^f A^{N-2}B & \dots & D^f \end{bmatrix}, \quad (3.42)$$

which is the system matrix of a blocked fat system with generic parameter matrices. From Zamani et al. [2011], it is well-known that (3.42) is full normal rank. Now consider the remaining rows of $M_1(Z)$, i.e. the matrix

$$\Pi = \begin{bmatrix} C^f A^{N-1} & C^f A^{N-2}B & \dots & C^f B & D^f \\ C^s A^{N-1} & C^s A^{N-2}B & \dots & C^s B & D^s \end{bmatrix}$$

which can be factorized as

$$\Pi = \begin{bmatrix} C^f & D^f \\ C^s & D^s \end{bmatrix} \begin{bmatrix} A^{N-1} & A^{N-2}B & \dots & B & 0 \\ 0 & 0 & \dots & 0 & I_m \end{bmatrix} \triangleq H\bar{\mathcal{R}}.$$

Since A is full rank, then also A^{N-1} is full rank and thus the matrix $\bar{\mathcal{R}}$ is full-row rank, namely $n + m$. Thus, the rank of Π depends on the rank of H , which, for generic choice of matrices C^s, D^s, C^f, D^f , is equal to $\alpha \triangleq \min\{p_f + p_s, m + n\}$. Then $\text{normal rank}(M_1(Z)) \leq n + (N - 1)p_f + \alpha$. However, since for the condition of tallness $p_s > N(m - p_f)$ and by assumption $n < (N - 1)(m - p_f)$, we have $n + m < (N - 1)(m - p_f) + m = N(m - p_f) + p_f < p_s + p_f$, and so $\alpha = n + m$. Hence $\text{normal rank}(M_1(Z)) \leq (N - 1)p_f + m + 2n$. Combining this bound with the lower bound found previously, we conclude that $\text{normal rank}(M_1(Z)) = (N - 1)p_f + m + 2n$.

On the Zero-freeness of Tall Multirate Systems with Coprime Output Rates

Abstract

This chapter explores discrete-time linear systems with multirate outputs, assuming that two measured output streams are available at coprime rates. These systems are examined in the literature in their blocked form. Hence, we focus on the several fundamental properties of blocked systems resulting from the blocking of multirate systems with coprime output rates. In particular, structural properties are studied and it is demonstrated that under a generic setting i.e. for a generic choice of parameter matrices of unblocked system, the blocked systems are minimal. Furthermore, zeros of tall blocked systems when the associated system matrix attains full-column rank, are examined in this chapter.

4.1 Introduction

In the field of econometric modeling, it is common to have situations under which some data are collected monthly, while other data may be obtained quarterly or even annually Mariano and Murasawa [2003], Banbura and Rünstler [2011]. It is also the case to have some data aggregated bimonthly and other data obtained quarterly Bass and Leone [1983]. The mentioned scenarios can be studied under multirate systems analysis. The former scenario is associated with multirate systems whose measured outputs have two parts, one available at all times (*fast outputs*) and the other one is available every N time instants (*slow outputs*). This type of multirate systems was the subject of study in the previous chapter. In this chapter, we refer to such systems as **multirate systems type-1**; in partial contrast, the second scenario corresponds to multirate systems whose measured outputs still have two parts, fast outputs and slow outputs, but, unlike the former case the rates of availability of the fast outputs and the slow outputs are coprime integers. Here, the term **multirate systems type-2**

is used to refer to these systems.

In this chapter, we focus on structural properties and zeros of type-2 systems. The analysis of zeros was done in Chapter 3 for multirate systems type-1. However, it turns out that the examination of zeros for type-2 systems is more complicated compared to that of type-1 systems. This is because type-1 systems are an especial class of type-2 systems.

In the rest of the current chapter, the term multirate systems is used to refer to multirate systems type-2 unless otherwise mentioned. In this chapter, we only focus on situations under which the associated blocked system has the matrix with full-column normal rank. Then under these conditions, it is shown that multirate systems with generic parameter matrices have no finite nonzero zeros.

This chapter is structured as follows. Section 4.2 introduces the problem under study. Then the idea of two-step blocking is introduced in Section 4.3 and based on that the reachability and observability of the blocked system associated with a multirate system are described in Section 4.2. In Section 4.4, the dynamic properties and particularly zeros of tall blocked systems are investigated. Finally, Section 4.5 provides concluding remarks.

4.2 Problem Formulation

In this section, first the formulation of the problem under study is given. The dynamics of an underlying system are defined by

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, \\ y_t &= Cx_t + Du_t, \end{aligned} \tag{4.1}$$

where $x_t \in \mathbb{R}^n$ is the state, $y_t \in \mathbb{R}^p$ the output, and $u_t \in \mathbb{R}^m$ the input. For this system, y_t exists for all t , and can in principle be measured at every time t . However, we are specifically interested in the situation where y_t exists for all t , but not all of its entries are measured at every time instant. In particular, the case where y_t has components that are observed at different rates, is considered. Indeed, it is assumed that there are two output streams, one available every t_f time instants and the other every t_s time instants, with t_f and t_s coprime integers. Without loss of generality, it is adopted that $t_f < t_s$. The noncoprime case is not treated here, since we believe that extensions of the ideas of this chapter or that of the previous chapter can tackle such a problem.

Without loss of generality, y_t can be decomposed as

$$y_t = \begin{bmatrix} y_t^f \\ y_t^s \end{bmatrix},$$

where $y_t^f \in \mathbb{R}^{p_f}$ is observed at $t = 0, t_f, 2t_f, \dots$, the fast part, and $y_t^s \in \mathbb{R}^{p_s}$ is observed at $t = 0, t_s, 2t_s, \dots$, the slow part; also $p_f > 0$, $p_s > 0$ and $p_f + p_s = p$. Correspondingly, matrices C and D can be expressed as

$$C = \begin{bmatrix} C^f \\ C^s \end{bmatrix}, \quad D = \begin{bmatrix} D^f \\ D^s \end{bmatrix}.$$

Thus, the multirate linear system (which we denote by Σ) corresponding to what is measured has the following dynamics:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t, \quad t = 0, 1, 2, \dots, \\ y_t^f &= C^f x_t + D^f u_t, \quad t = 0, t_f, 2t_f, \dots, \\ y_t^s &= C^s x_t + D^s u_t, \quad t = 0, t_s, 2t_s, \dots \end{aligned} \quad (4.2)$$

Since the rates of availability of fast outputs and slow outputs are assumed to be coprime, in order to obtain a blocked linear time-invariant system associated with the system (4.2), one has to block the system by the rate $N \triangleq t_f t_s$. Actually, there exist N distinct ways to block the above multirate system. All N resultant blocked systems share common poles properties but their zeros might not be identical. The distinction is made apparent below.

For the tag point $\tau \in \{1, 2, \dots, N\}$, define

$$U_t^\tau \triangleq \begin{bmatrix} u_{t+\tau} \\ u_{t+\tau+1} \\ \vdots \\ u_{t+\tau+N-1} \end{bmatrix}, \quad Y_t^\tau \triangleq \begin{bmatrix} y_{t+\tau+\theta_f}^f \\ y_{t+\tau+\theta_f+t_f}^f \\ y_{t+\tau+\theta_f+2t_f}^f \\ \vdots \\ y_{t+\tau+\theta_f+(t_s-1)t_f}^f \\ y_{t+\tau+\theta_s}^s \\ y_{t+\tau+\theta_s+t_s}^s \\ y_{t+\tau+\theta_s+2t_s}^s \\ \vdots \\ y_{t+\tau+\theta_s+(t_f-1)t_s}^s \end{bmatrix}, \quad (4.3)$$

where $t = 0, N, 2N, \dots$, $\theta_f \triangleq (N - \tau) \bmod t_f$ and $\theta_s \triangleq (N - \tau) \bmod t_s$. Finally $x_t^\tau \triangleq x_{t+\tau}$.

The integer θ_f (θ_s) admits a physical interpretation as the delay between the tag point τ and the first among the following time instants in which a sample of y_t^f (y_t^s) is available. Figure 4.1 shows an example for $t_f = 2$, $t_s = 3$ and $\tau = 1$, which results in $\theta_f = 1$, $\theta_s = 2$.

The blocked system Σ_τ is defined by

$$\begin{aligned} x_{t+N}^\tau &= A_\tau x_t^\tau + B_\tau U_t^\tau, \\ Y_t^\tau &= C_\tau x_t^\tau + D_\tau U_t^\tau, \end{aligned} \quad (4.4)$$

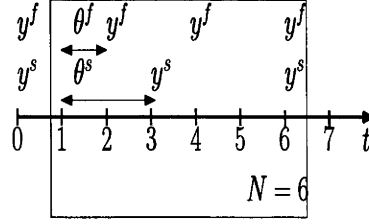


Figure 4.1: Collection of observations in a blocked system. The input signal u is assumed to be available at every time instant.

where

$$\begin{aligned}
 A_\tau &\triangleq A^N, \\
 B_\tau &\triangleq [A^{N-1}B \quad A^{N-2}B \quad \dots \quad AB \quad B], \\
 C_\tau &\triangleq \begin{bmatrix} C_\tau^f \\ C_\tau^s \end{bmatrix}, \\
 C_\tau^f &\triangleq [(C^f A^{\theta^f})^T (C^f A^{\theta^f+t_f})^T \dots (C^f A^{\theta^f+N-t_f})^T]^T, \\
 C_\tau^s &\triangleq [(C^s A^{\theta^s})^T (C^s A^{\theta^s+t_s})^T \dots (C^s A^{\theta^s+N-t_s})^T]^T, \\
 D_\tau &\triangleq \begin{bmatrix} D_\tau^f \\ D_\tau^s \end{bmatrix}.
 \end{aligned} \tag{4.5}$$

Furthermore,

$$D_{\tau[i,j]}^f = Q^f(\theta^f + (i-1)t_f + (j-1)), \tag{4.6}$$

where $D_{\tau[i,j]}^f$ denotes the $p_f \times m$ block of matrix D_τ^f in block position i, j and

$$Q^f(v) = \begin{cases} 0 & v < 0, \\ D^f & v = 0, \\ C^f A^{v-1}B & v > 0. \end{cases} \tag{4.7}$$

The matrix D_τ^s can be defined similarly to (4.6), replacing the parameters θ^f, t_f with θ^s, t_s , respectively.

The following example suggests a possible structure of the blocked system Σ_τ .

Example 4.2.1. Consider the system

$$\begin{aligned}
 x_{t+1} &= ax_t + bu_t, \quad t = 0, 1, 2, \dots, \\
 y_t^f &= c^f x_t + d^f u_t, \quad t = 0, t_f, 2t_f, \dots, \\
 y_t^s &= c^s x_t + d^s u_t, \quad t = 0, t_s, 2t_s, \dots
 \end{aligned} \tag{4.8}$$

where the $a, b, c^f, d^f \in \mathbb{R}$ and $C^s = [c_1^s c_2^s]^T, D^s = [d_1^s d_2^s]^T \in \mathbb{R}^2$. We assume the same values for t_f and t_s as in Fig. 4.1 i.e. $t_f = 2$ and $t_s = 3$, so that the blocking rate is $N = 6$. Then, for $\tau = 1$ the blocked system Σ_τ is described by the matrices

$$\begin{aligned}
 A_\tau &= a^6, \\
 B_\tau &= [a^5b \ a^4b \ a^3b \ a^2b \ ab \ b], \\
 C_\tau^f &= [c^f a \ c^f a^3 \ c^f a^5]^T, \\
 C_\tau^s &= [c_1^s a^2 \ c_2^s a^2 \ c_1^s a^5 \ c_2^s a^5]^T, \\
 D_\tau^f &= \begin{bmatrix} c^f b & d^f & 0 & 0 & 0 & 0 \\ c^f a^2 b & c^f a b & c^f b & d^f & 0 & 0 \\ c^f a^4 b & c^f a^3 b & c^f a^2 b & c^f a b & c^f b & d^f \end{bmatrix}, \\
 D_\tau^s &= \begin{bmatrix} c_1^s a b & c_1^s b & d_1^s & 0 & 0 & 0 \\ c_2^s a b & c_2^s b & d_2^s & 0 & 0 & 0 \\ c_1^s a^4 b & c_1^s a^3 b & c_1^s a^2 b & c_1^s a b & c_1^s b & d_1^s \\ c_2^s a^4 b & c_2^s a^3 b & c_2^s a^2 b & c_2^s a b & c_2^s b & d_2^s \end{bmatrix}.
 \end{aligned} \tag{4.9}$$

4.3 Structural Properties

4.3.1 An Interpretation in Terms of Two-step Blocking

In the previous section, the structure of the blocked system Σ_τ has been provided. The presence of two measured output streams available at coprime rates seems to induce a complex structure for Σ_τ . In this section, it is first demonstrated that the system Σ_τ admits a natural interpretation in terms of a two-step blocking once the fast and slow rate outputs are considered separately. The main idea is to split Σ into the systems $\Sigma^f \triangleq (A, B, C^f, D^f)$ and $\Sigma^s \triangleq (A, B, C^s, D^s)$ and then to separately block these systems (according to a certain procedure which will be discussed in more detail later). The two resultant blocked systems evidently share the same input-state dynamics which is identical to the input-state dynamics of the system Σ_τ . Furthermore, one can construct the output dynamics of the system Σ_τ by simply stacking the output dynamics of the blocked system obtained only by considering the fast rate outputs over those obtained by only considering the slow rate outputs. To illustrate the approach in more details, we first focus on Σ^f only; it is explained that the system $\tilde{\Sigma}_\tau^f \triangleq (A_\tau, B_\tau, C_\tau^f, D_\tau^f)$, which is the part of the blocked final system Σ_τ associated with the fast rate output only, can be obtained by the following operations.

1. Block the system Σ^f with $\tau_f = t_f - \theta_f$ replacing the tag point τ and the input block size of t_f (**step 1**).
2. Block the system resulting from the previous step, call it Σ_{τ_f} , with the input and the output block size of t_s and the tag point equal to zero (**step 2**).

It is worthwhile remarking that one can easily obtain the system $\tilde{\Sigma}_\tau^s \triangleq (A_\tau, B_\tau, C_\tau^s, D_\tau^s)$ by blocking Σ^s with the tag point equal to $\tau_s = t_s - \theta_s$ and input block size of t_s . Then blocking the system obtained from the previous step, say Σ_{τ_s} , by the input and output block size of t_f and the tag point equal to zero. Finally, the system Σ_τ is obtained by considering A_τ and B_τ and simply stacking the matrices C_τ^f over C_τ^s and D_τ^f over D_τ^s , respectively. For the sake of complete explanation, it is now demonstrated how A_τ, B_τ, C_τ^f and D_τ^f can be obtained through the two-step blocking approach.

4.3.1.1 Step 1

The result of step 1 of the blocking procedure is the system Σ_{τ_f} , which is defined by a quadruple $(A_{\tau_f}, B_{\tau_f}, C_{\tau_f}, D_{\tau_f})$, where

$$\begin{aligned} A_{\tau_f} &\triangleq A^{t_f}, & B_{\tau_f} &\triangleq \begin{bmatrix} A^{(t_f-1)}B & \dots & AB & B \end{bmatrix}, \\ C_{\tau_f} &\triangleq C^f A^{\theta_f}, \\ D_{\tau_f} &\triangleq \begin{bmatrix} C^f A^{\theta_f-1}B & C^f A^{\theta_f-2}B & \dots & D^f & 0_{p_f \times m(t_f-\theta_f-1)} \end{bmatrix}. \end{aligned} \quad (4.10)$$

4.3.1.2 Step 2

In this step, one performs blocking on the system Σ_{τ_f} . It is shown in the following that the resulting system is exactly the system $\tilde{\Sigma}_\tau^f$. One can easily observe that when the system Σ_{τ_f} is blocked with input block size of t_s , the state matrix of the resultant system will be in the form $A_{\tau_f}^{t_s} = A^{t_s t_f} = A_\tau$. Furthermore, the input matrix of the resultant system has the following structure

$$\begin{bmatrix} A_{\tau_f}^{(t_s-1)} B_{\tau_f} & A_{\tau_f}^{(t_s-2)} B_{\tau_f} & \dots & A_{\tau_f} B_{\tau_f} & B_{\tau_f} \end{bmatrix}. \quad (4.11)$$

Using simple algebra computations one can see that

$$\begin{aligned} A_{\tau_f}^{(t_s-i)} B_{\tau_f} &= A^{t_f(t_s-i)} \begin{bmatrix} A^{(t_f-1)}B & A^{(t_f-2)}B & \dots & B \end{bmatrix} \\ &= \begin{bmatrix} A^{(N-(i-1)t_f-1)}B & A^{(N-(i-1)t_f-2)}B & \dots \\ \dots & A^{(N-it_f+1)}B & A^{(N-it_f)}B \end{bmatrix}, \end{aligned} \quad (4.12)$$

$$\begin{aligned} A_{\tau_f}^{(t_s-i)} B_{\tau_f} &= A^{t_f(t_s-i)} \begin{bmatrix} A^{(t_f-1)}B & A^{(t_f-2)}B & \dots & B \end{bmatrix} \\ &= \begin{bmatrix} A^{(N-(i-1)t_f-1)}B & A^{(N-(i-1)t_f-2)}B & \dots \\ \dots & A^{(N-it_f+1)}B & A^{(N-it_f)}B \end{bmatrix}, \end{aligned} \quad (4.13)$$

and it is straightforward to see that the resultant matrix is B_τ . Moreover, the output matrix of the resulting blocked system is

$$\begin{bmatrix} C_{\tau_f}^T & (C_{\tau_f} A_{\tau_f})^T & (C_{\tau_f} A_{\tau_f}^2)^T & \dots & (C_{\tau_f} A_{\tau_f}^{t_s-1})^T \end{bmatrix}^T. \quad (4.14)$$

Again, one can see that $C_{\tau_f} A_{\tau_f}^i = C^f A^{\theta_f + i t_f}$, hence the resulting matrix is C_τ^f . Finally, the direct feedthrough matrix of the blocked system is

$$\begin{bmatrix} D_{\tau_f} & 0 & \dots & 0 \\ C_{\tau_f} B_{\tau_f} & D_{\tau_f} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ C_{\tau_f} A_{\tau_f}^{t_s-2} B_{\tau_f} & C_{\tau_f} A_{\tau_f}^{t_s-3} B_{\tau_f} & \dots & D_{\tau_f} \end{bmatrix} \quad (4.15)$$

and by substituting parameters from (4.10) one can easily verify that the matrix in (4.15) is D_τ^f .

Hence, the dynamics of the fast rate outputs of the blocked system Σ_τ can be obtained from the two-step blocking process described above. One can readily observe that the slow rate outputs of the blocked system Σ_τ can be obtained in a similar way.

4.3.2 Minimality of the Blocked System

In this section, the reachability¹ and observability properties of the blocked system Σ_τ given corresponding properties for the unblocked multirate system are discussed. To this end, the following result from Colaneri and Longhi [1995] must be recalled.

Lemma 4.3.1. *Colaneri and Longhi [1995] The unblocked multirate system (4.2) is reachable (observable) at time τ if and only if the system Σ_τ is reachable (observable).*

With the help of the above lemma, the following result proves the minimality of the system Σ_τ for a set of generic parameter matrices A, B , etc.

Theorem 4.3.2. *Consider the system Σ_τ defined by the quadruple $\{A_\tau, B_\tau, C_\tau, D_\tau\}$, where $A_\tau, B_\tau, C_\tau, D_\tau$ are specified by (4.5). Then, for a generic choice of the matrices A, B, C^f, D^f, C^s and D^s the system Σ_τ is reachable and observable (and so minimal).*

Proof. The proof is straightforward. Since the parameter matrices A, B, C^f, C^s, D^f and D^s assume generic values the associated multirate dynamics is both reachable and observable at time τ . Then using the result of Lemma 4.3.1, the same should hold for Σ_τ . \square

Remark 4.3.3. *It is immediate that the systems Σ_{τ_f} and Σ_{τ_s} are generically reachable and observable.*

¹It is important to mention that the notions of reachability and controllability are equivalent when the state matrix is nonsingular.

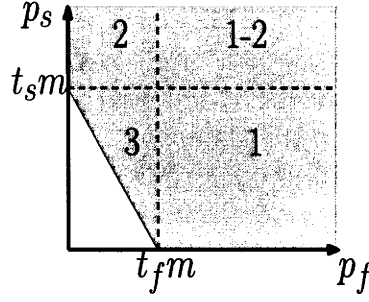


Figure 4.2: Partitioning of the condition for tallness.

4.4 Zeros of the Blocked System

In this section, zeros of Σ_τ are analyzed. While poles are well known Colaneri and Longhi [1995], Bittanti [1986] and Bittanti and Colaneri [2009], the problem of verifying the possible presence of zeros is nontrivial and needs a deep analysis. In particular, motivated by applications in econometric modeling and systems and control, tall systems are considered. The general condition for the system Σ_τ to be tall is

$$t_s p_f + t_f p_s > Nm. \quad (4.16)$$

Such a relation can be interpreted as a subset of the first orthant of the plane (p_f, p_s) ; for example it can be partitioned as

1. $p_f > t_f m$;
2. $p_s > t_s m$;
3. $\frac{Nm - t_f p_s}{t_s} < p_f \leq t_f m, p_s \leq t_s m$.

These partitions are depicted in Fig. 4.2.

In the rest of this chapter, finite nonzero zeros are investigated. Infinite zeros and zeros at the origin certainly require separate techniques for the examination in the case $t_f = 1$ treated in Chapter 3, and it is highly likely to be the case here too. It is worthwhile mentioning that due to the introduction of several new parameters e.g. t_f, t_s, τ_f, τ_s etc., the analysis of zeros at infinity and the origin in this case is more intricate compared to the case discussed in the previous chapter. Thus, infinite zeros and zeros at the origin will be considered in future. Further, we restrict our attention to regions one and two in Fig. 4.2. It is worthwhile remarking that the region three is associated with tall blocked systems with fast and slow parts neither of which alone is tall. The analysis of zeros for this region appear to be harder compared to the other regions. In particular, it is shown that for a generic choice of the parameter matrices A, B , etc., the blocked system Σ_τ has no finite nonzero zeros provided that their state space dimension satisfies certain inequalities. The object of our interest is

the so-called system matrix Σ_τ , defined as the $(t_s p_f + t_f p_s + n) \times (n + Nm)$ matrix

$$M_\tau(Z) \triangleq \begin{bmatrix} ZI - A_\tau & -B_\tau \\ C_\tau & D_\tau \end{bmatrix}, \quad Z \in \mathbb{C}, \quad (4.17)$$

since such a matrix is of paramount importance for the study of the zeros of a system (see Definition 2.3.2).

Now let $S_\tau(\zeta)$, where ζ is both a complex variable and t_f -steps forward shift operator, denote the system matrix corresponding to the system Σ_{τ_f} .

Here, to prove the main result we follow the same approach as in the previous chapter. In the following, we start our analysis by focusing on Σ_{τ_f} and considering the value $\tau_f = 1$, which corresponds to $\theta_f = t_f - 1$. First, we need to define a square submatrix of $S_1(\zeta)$, call it $N_1(\zeta)$, such that $\text{normal rank}(N_1(\zeta)) = \text{normal rank}(S_1(\zeta))$. Then

$$S_1(\zeta) = \begin{bmatrix} N_1(\zeta) \\ C_2 \quad D_2 \end{bmatrix}, \quad (4.18)$$

where C_2 and D_2 capture those rows of C_{τ_f} and D_{τ_f} that are not included in $N_1(\zeta)$.

Proofs for the the next two results are very similar to those of the Proposition 3.2.10 and Theorem 3.2.12, accordingly and hence omitted.

Proposition 4.4.1. *Let the matrix $N_1(\zeta)$ be a submatrix of $S_1(\zeta)$ formed via the procedure described. Then for generic values of the matrices A, B, C^f, D^f with $p_f > t_f m$, for any finite ζ_0 for which the matrix $N_1(\zeta_0)$ has less rank than the normal rank of $N_1(\zeta)$, its rank is one less than its normal rank.*

The next theorem shows that if the two-step blocking procedure is used to construct Σ_τ , then after the first step, there is freedom from finite nonzero zeros generically.

Theorem 4.4.2. *For a generic choice of the matrices $\{A, B, C^f, D^f\}$ with $p_f > t_f m$, the system matrix $S_{\tau_f}(\zeta)$, has rank equal to its normal rank for all finite nonzero values or equivalently the system Σ_{τ_f} has no finite nonzero zeros.*

The following result is recalled from Chapter 2.

Lemma 4.4.3. *The system $\tilde{\Sigma}_\tau^f$ has a zero if and only if the system Σ_{τ_f} has a zero.*

Now with the help of Lemma 4.4.3 and Theorem 4.4.2, the following corollary becomes immediate.

Corollary 4.4.4. *For a generic choice of a quadruple $\{A, B, C^f, D^f\}$, with $p_f > t_f m$, the blocked system $\tilde{\Sigma}_\tau^f$ has no finite nonzero zeros.*

Then one can easily observe that the following result also holds.

Corollary 4.4.5. *For a generic choice of a quadruple $\{A, B, C^s, D^s\}$, with $p_s > t_s m$, the blocked system $\tilde{\Sigma}_\tau^s$ has no finite nonzero zeros.*

Our ultimate interest is to study the zero-freeness of Σ_τ . Note that according to Definition 2.3.2 the normal rank of system matrix $M_\tau(Z)$ plays an important role in the zero properties of Σ_τ . The proof of zero-freeness for the system Σ_τ is much simpler when the associated system matrix has full-column normal rank compared to the case where it has less than full-column normal rank. Here, we deal with the simpler problem before tackling the harder one in the future. In the following, sufficient conditions for $M_\tau(Z)$ to have full-column normal rank are provided. To achieve this, we first focus on the normal rank of $S_{\tau_f}(\zeta)$.

The following proposition even though restricted to a particular choice of τ_f , will help us later to provide sufficient conditions for all $S_{\tau_f}(\zeta)$ to have full-column normal rank.

Proposition 4.4.6. *Consider a generic choice of the matrices $\{A, B, C^f, D^f\}$. Assume that $n \geq (t_f - 1)m$, $p_f > t_f m$; then the matrix D_{τ_f} , for $\tau_f = 1$, has full-column rank.*

Proof. First, observe that when $\tau_f = 1$, we have $\theta_f = t_f - 1$. Furthermore, the matrix D_{τ_f} admits the factorization, $[C^f \ \dots \ C^f \ D^f] \Gamma$, and $\Gamma := \text{diag}\{A^{\theta_f-1}B, A^{\theta_f-2}B, \dots, B, I_m\}$. Since $n \geq m$, Γ is not full-row rank. Note that the matrix A is generic and thus is nonsingular; hence, one can select $n - m$ rows from each block $A^{\theta_f-1}B, A^{\theta_f-2}B, \dots, B, I_m$ and discard them in order to obtain a matrix, say $\tilde{\Gamma}$, which becomes full-row rank, namely $(\theta_f + 1)m = t_f m$. Similarly, one can construct the matrix $\tilde{\Pi}$, which is defined by discarding the corresponding columns from the matrix $[C^f \ \dots \ C^f \ D^f]$. Due to the genericity of matrix A , the selection process can be done such that the matrix $\tilde{\Pi}$ includes all the columns of the matrix C^f (and their linear combinations) and all columns of matrix D^f . Since $\tilde{\Gamma}$ is full-row rank, the rank of D_{τ_f} only depends on the rank of $\tilde{\Pi}$. Moreover, one can easily observe that the rank of $\tilde{\Pi}$ is equal to the rank of the matrix $\tilde{\Pi} := [C^f \ D^f] \in \mathbb{R}^{p_f \times [n+m]}$. Under the assumption $n \geq (t_f - 1)m$, both the integers p_f and $n + m$ are greater than or equal to the number of columns of D_{τ_f} , namely $t_f m$, hence D_{τ_f} is full-column rank. \square

Remark 4.4.7. *In the above proposition, for the particular choice $\tau_f = 1$, we provided sufficient conditions for D_{τ_f} to have full-column rank. This does not hold for all values of τ_f . The latter is due to the fact that depending on the choice of τ , the highest time index of input sequences appearing in the blocked input vector becomes strictly larger than that of output sequences appearing in the blocked output vector.*

Even though the conclusion of Proposition 4.4.6 is restricted to a particular choice of τ_f , we now deliver the following lemma which holds for all $S_{\tau_f}(\zeta)$.

Lemma 4.4.8. *Consider the system Σ_{τ_f} with a generic choice of the matrices $\{A, B, C^f, D^f\}$. Assume that $n \geq (t_f - 1)m$, $p_f > t_f m$; then the system matrix $S_{\tau_f}(\zeta)$ has full-column normal rank for any value of $\tau_f \in \{1, \dots, t_f\}$.*

Proof. It was demonstrated in Proposition 4.4.6 that for $\tau_f = 1$ the feedthrough matrix has full-column rank. Then with the help of proof of Lemma 3.2.2, one can conclude the associated system matrix has full-column normal rank. Moreover, note that all

$S_{\tau_f}(\zeta)$, for all possible values of τ_f , have the same normal rank (see Proposition 3.4.3). Hence, under the conditions established in the statement of the lemma, $S_{\tau_f}(\zeta)$ has full-column normal rank. \square

From Lemma 4.4.8 and Theorem 4.4.2, the following proposition which has a trivial proof, is introduced.

Proposition 4.4.9. *Consider the system Σ_{τ_f} with a generic choice of the matrices $\{A, B, C^f, D^f\}$. Assume that $n \geq (t_f - 1)m$, $p_f > t_fm$; then the system matrix $S_{\tau_f}(\zeta)$ has full-column rank for all ζ .*

The main result of this chapter about zero-freeness of the system Σ_τ is provided in the next theorem.

Theorem 4.4.10. *Consider the system Σ_τ defined by the sextuple $\{A, B, C^f, D^f, C^s, D^s\}$ with generic values for the entries of the defining matrices. Assume that $p_f > t_fm$ and $n \geq (t_f - 1)m$; then the system matrix $M_\tau(Z)$ has full-column rank for all $Z \in \mathbb{C} - \{0, \infty\}$, and accordingly the system Σ_τ has no finite nonzero zeros.*

Proof. Recall from Corollary 2.3.6 that the system Σ_{τ_f} has full-column normal rank if and only if the associated system matrix of $\tilde{\Sigma}_{\tau_f}^f$ has full-column normal rank. Then, with the help of Lemma 4.4.3 and Proposition 4.4.9, one can easily conclude that the system matrix of $\tilde{\Sigma}_{\tau_f}^f$ has full-column rank for all finite nonzero values. Hence, it is immediate that $M_\tau(Z)$ has full-column rank for all finite nonzero values as well. \square

Remark 4.4.11. *In Chapter 3, where $t_f = 1$, there was no parallel of the restriction on the state dimension. If we set $t_f = 1$ in Theorem 4.4.10, the condition simply becomes one of saying the state vector has dimension at least 0, which corresponds to giving no conditions on the state dimension.*

As a parallel of the above theorem the preceding theorem can be evidently stated.

Theorem 4.4.12. *Consider the system Σ_τ defined by the sextuple $\{A, B, C^f, D^f, C^s, D^s\}$ with generic values for the entries of the defining matrices. Assume that $p_s > t_sm$ and $n \geq (t_s - 1)m$; then the system matrix $M_\tau(Z)$ has full-column rank for all $Z \in \mathbb{C} - \{0, \infty\}$, and accordingly the system Σ_τ has no finite nonzero zeros.*

The following numerical example is introduced to further illustrate the earlier results and also show gaps which can be subjects for future research.

Example 4.4.13. *In this example, we examine zeros of the system Σ_τ for three different set of parameters. We first suppose that $m = 4$, $p_f = 9$, $p_s = 9$, $n = 9$, $t_f = 2$ and $t_s = 3$; we refer to this set as 'set 1'. It is worthwhile mentioning that the values in set 1 satisfy the conditions stated in the statement of Theorem 4.4.10. It can be verified that for generic set of parameter matrices A, B, C^f, D^f, C^s and D^s the system Σ_τ has no finite nonzero zeros. This supports the result of Theorem 4.4.10. Furthermore, the following table records zeros of the system Σ_τ for different values of τ .*

Table 4.1: Zeros at the origin and infinity for the system in Example 4.4.13- set 1

Value of τ \ number of zeros	at the origin	at infinity
1	4	0
2	0	4
3	0	0
4	0	0
5	4	0
6	0	4

Let us now consider the same values as in set 1 except for $n = 3$, we name this case 'set 2'. It is obvious that the state space dimension i.e. n , does not fulfill the condition of Theorem 4.4.10. However, it can be checked that again for a generic choice of matrices A, B , etc., the system Σ_τ has no finite nonzero zeros. Furthermore, the associated system matrix has normal rank equal to 25 i.e. the system matrix has less than full-column normal rank. Moreover, those zeros at infinity and at the origin follow the table below.

Table 4.2: Zeros at the origin and infinity for the system in Example 4.4.13- set 2

Value of τ \ number of zeros	at the origin	at infinity
1	3	0
2	0	3
3	0	0
4	0	0
5	3	0
6	0	3

Finally, we assume a set which is concerned with region three in Fig. 4.2. Hence, the parameters are set to be $m = 4$, $p_f = 5$, $p_s = 9$, $n = 7$, $t_f = 2$, $t_s = 3$ (we call this 'set 3'). One can test that for a generic choice of parameter matrices A, B , etc., the system matrix has full-column rank for all finite nonzero value of Z . Furthermore, it is also easy to check for zeros at the origin and infinity. This is recorded in Table 4.3.

Remark 4.4.14. The above numerical example suggests that the result of Theorem 4.4.10 is not confined to region one in Fig. 4.2 and it holds in general. Furthermore, in relation to zeros at the origin and infinity, one can also conjecture that Lemma 3.3.1 can be applied to this case as well.

Table 4.3: Zeros at the origin and infinity for the system in Example 4.4.13- set 3

Value of τ \ number of zeros	at the origin	at infinity
1	4	0
2	0	4
3	3	0
4	0	3
5	4	0
6	0	4

4.5 Summary

The linear multirate systems with coprime output rates were discussed in this chapter. The procedure called two-step blocking was illustrated and with the help of that, structural properties of blocked systems associated with multirate systems were explored. It was shown that the blocked systems are generically observable and reachable. Furthermore, the finite nonzero zeros of blocked systems were analyzed. This was done only for those blocked systems having system matrix with full-column normal rank. It was shown that such systems generically have no finite nonzero zeros. The numerical results suggest that this holds in general i.e. even when the associated system matrix has less full-column normal rank. However, for generalization of the main result of this chapter i.e. Theorem 4.4.10, the technique used in the previous chapter fails because the direct feedthrough matrix is very structured and may not contain any free parameters. In relation to zeros at the origin and at infinity, numerical examples recommend that Lemma 3.3.1 also holds for this case. However, calculating number of zeros at infinity is difficult here as relations between several parameters like $\theta_f, \theta_s, t_f, t_s, n, p_f, p_s$ have to be considered for such a calculation.

On the Identifiability of Regular and Singular Multivariate Autoregressive Models from Mixed Frequency Data

Abstract

The identifiability of an underlying high frequency multivariate regular and singular autoregressive models from mixed frequency observations is studied in this chapter. It is particularly shown that the parameters of this model are generically identifiable from those population second order moments which can be observed in principle¹. Equivalently, such a model is identifiable on a superset of an open and dense subset of the parameter space. It is important to note if the identifiability exists, the parameters of the associated models i.e. the system and noise parameters, and thus all second moments of the output process can be estimated consistently from mixed frequency data. Then linear least squares methods for forecasting and interpolating nonobserved output variables can be applied.

5.1 Introduction

One way of handling mixed frequency data is the method of blocking, which is discussed in complete detail in the last three chapters and used in Zamani et al. [2011] and Ghysels [2012]. In chapters 3 and 4, it was clearly shown that a tall blocked linear time-invariant system derived from an underlying unblocked linear system with one or more missing outputs is generically zero-free. This chapter is built on the results of the two previous chapters; here, we postulate that there exists an autoregressive model operating at the highest sampling frequency, which is legitimate in the light of the results of chapters 3 and 4 which demonstrated that the associated blocked

¹Second order moments which are observed 'in principle' are those which can be consistently estimated from sample statistics when the number of samples goes to infinity.

systems are generically zero-free apart from possible choices of zeros at the origin or infinity.

It is worthwhile mentioning that the covariance matrix associated with input sequences of an autoregressive (AR) model can be either regular or singular. The terms *regular autoregressive model* and *singular autoregressive model* are used to refer to the former and latter, accordingly. This chapter discusses regular and singular autoregressive models. However, singular autoregressive models require a more detailed study; the next chapter explores these models in more details.

In this chapter, we propose the method of modified extended Yule-Walker equations. This is a modification of an earlier approach introduced in Chen and Zadrozny [1998]. We then use the proposed approach to study the identifiability of regular and singular AR systems. It is demonstrated that using the proposed approach one can generically identify the system and noise parameters from those population second moments which can be observed in principle.

This chapter is structured as follows. In the next section, we formulate the problem under study. Then in Section 5.3, we first briefly review Yule-Walker equations. Then based on that we introduce the modified extended Yule-Walker equations. This method enables us to show that systems and noise parameters are generically identifiable from available population second moments. Finally, Section 5.4 provides concluding remarks.

5.2 Problem Formulation

Consider the case where this high frequency system is a vector autoregression of order p , i.e.

$$\begin{aligned} y_t = \begin{bmatrix} y_t^f \\ y_t^s \end{bmatrix} &= \underbrace{\begin{bmatrix} a_{ff}(1) & a_{fs}(1) \\ a_{sf}(1) & a_{ss}(1) \end{bmatrix}}_{=A_1} \begin{bmatrix} y_{t-1}^f \\ y_{t-1}^s \end{bmatrix} + \cdots + \\ &+ \underbrace{\begin{bmatrix} a_{ff}(p) & a_{fs}(p) \\ a_{sf}(p) & a_{ss}(p) \end{bmatrix}}_{=A_p} \begin{bmatrix} y_{t-p}^f \\ y_{t-p}^s \end{bmatrix} + \underbrace{\begin{bmatrix} v_t^f \\ v_t^s \end{bmatrix}}_{=v_t}, \quad t \in \mathbb{Z}, \end{aligned} \quad (5.1)$$

where $a_{ff}(i) \in \mathbb{R}^{n_f \times n_f}$, $a_{ss}(i) \in \mathbb{R}^{n_s \times n_s}$, $a_{fs}(i) \in \mathbb{R}^{n_f \times n_s}$, and where n_f is the number of the components observed at highest frequency, n_s is the number of components observed only for $t \in N\mathbb{Z}$, i.e., every N th time point, and $n = n_f + n_s$. Throughout this chapter it is assumed that the high frequency system (5.1) is stable, and that we restrict ourselves to the steady state and thus stationary solution. Moreover, we

consider the case where the innovation variance $\Sigma = \mathbb{E} \left[\begin{bmatrix} v_t^f \\ v_t^s \end{bmatrix} \begin{bmatrix} (v_t^f)^T & (v_t^s)^T \end{bmatrix} \right] = \begin{bmatrix} \Sigma_{ff} & \Sigma_{fs} \\ \Sigma_{sf} & \Sigma_{ss} \end{bmatrix}$, where $\Sigma_{ff} \in \mathbb{R}^{n_f \times n_f}$, $\Sigma_{ss} \in \mathbb{R}^{n_s \times n_s}$, and $\Sigma_{fs} \in \mathbb{R}^{n_f \times n_s}$, is regular as

well as the case where this variance is singular. The singular case is important for generalized linear dynamic factor models (see Forni and Lippi [2001] and Filler et al. [2009]). In the singular case, when Σ is of rank $q < n$, one can write Σ as $\Sigma = bb^T$ where b is an $(n \times q)$ matrix. Accordingly, $v_t = b\varepsilon_t$, where $\mathbb{E}[\varepsilon_t \varepsilon_t^T] = I_q$. For given Σ , b is unique up to postmultiplication by an orthogonal matrix and thus b can be made unique. Whereas in the regular case Σ is parameterized by its on and above diagonal elements, in the singular case b can be used for the parameterization of Σ , because its free parameters are more easily seen in this way. Even though the results of this section can be applied to the singular case, this case needs further detailed study and is discussed in more detail in the next chapter.

System (5.1) can be written in block companion form as

$$\underbrace{\begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix}}_{=x_{t+1}} = \underbrace{\begin{bmatrix} A_1 & \cdots & A_{p-1} & A_p \\ I_n & & & \\ & \ddots & & \\ & & I_n & 0 \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{bmatrix}}_{=x_t} + \underbrace{\begin{bmatrix} b \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=\mathcal{B}} \varepsilon_t. \quad (5.2)$$

The Lyapunov equation, where $\Gamma = \mathbb{E}[x_t x_t^T]$, for the system (5.2) is

$$\Gamma - \mathcal{A}\Gamma\mathcal{A}^T = \mathcal{B}\mathcal{B}^T. \quad (5.3)$$

The results of this chapter cover regular AR systems. They are also applicable to singular AR systems when the matrix Γ is nonsingular. The situation where Γ is singular needs more attention and is studied in the next chapter.

The central problem considered in this chapter is identifiability, i.e. whether $\begin{bmatrix} a_{ff}(i) & a_{fs}(i) \\ a_{sf}(i) & a_{ss}(i) \end{bmatrix}$, $i \in \{1, \dots, p\}$ and Σ are uniquely identified from those population second moments which can be observed in principle, i.e. $\gamma^{ff}(h) = \mathbb{E} \left[y_{t+h}^f [y_t^f]^T \right]$, $h \in \mathbb{Z}$; $\gamma^{fs}(h) = \mathbb{E} \left[y_{t+h}^f [y_t^s]^T \right]$, $h \in \mathbb{Z}$; $\gamma^{ss}(h) = \mathbb{E} \left[y_{t+h}^s [y_t^s]^T \right]$, $h \in \mathbb{N}\mathbb{Z}$. Note that if identifiability holds and there is an available algorithm to compute the parameters from the observed second moments one can reconstruct the missing moments $\gamma^{ss}(h) = \mathbb{E} \left[y_{t+h}^s [y_t^s]^T \right]$, $h \in \mathbb{N}\mathbb{Z} - j$; $j = 1, \dots, N-1$ and then linear least squares methods for forecasting and interpolating nonobserved output variables can be applied. In other words, this identifiability is an important step in getting consistent estimators of the system and noise parameters and thus of the missing second moments $\gamma^{ss}(h)$, $h \in \mathbb{N}\mathbb{Z} - j$; $j = 1, \dots, N-1$ based on the mixed frequency data available.

5.3 Modified Extended Yule-Walker Equations

In this section, the extended Yule-Walker equations as proposed by Chen and Zadrozny [1998] are modified. Then based on the achieved modification, the central results of this chapter are provided. In particular, it is demonstrated that for generic AR(p) systems the modified extended Yule-Walker equations have a unique solution. Thus, one can generically have identifiability. This in turn implies that the system and noise parameters can be estimated consistently under an assumption guaranteeing the consistent estimation of those population second moments, which can be observed in principle. The result given holds for regular as well as for singular AR systems.

5.3.1 Derivation of the Modified Extended Yule-Walker Equations for Mixed Frequency Data

By postmultiplying equation (5.1) by y_{t-j}^T , $j > 0$ and forming expectations, one can obtain the extended Yule-Walker equations

$$\begin{aligned} & \left[\begin{array}{cc|ccc} \gamma^{ff}(1) & \gamma^{fs}(1) & \cdots & \gamma^{ff}(p) & \gamma^{fs}(p) & \cdots \\ \gamma^{sf}(1) & \gamma^{ss}(1) & \cdots & \gamma^{sf}(p) & \gamma^{ss}(p) & \cdots \end{array} \right] = \\ & \left[\begin{array}{cc|ccc} a_{ff}(1) & a_{fs}(1) & \cdots & a_{ff}(p) & a_{fs}(p) \\ a_{sf}(1) & a_{ss}(1) & \cdots & a_{sf}(p) & a_{ss}(p) \end{array} \right] \times \\ & \left[\begin{array}{cc|ccc} \gamma^{ff}(0) & \gamma^{fs}(0) & \cdots & \gamma^{ff}(p-1) & \gamma^{fs}(p-1) & \cdots \\ \gamma^{sf}(0) & \gamma^{ss}(0) & \cdots & \gamma^{sf}(p-1) & \gamma^{ss}(p-1) & \cdots \\ \hline \gamma^{ff}(1-p) & \gamma^{fs}(1-p) & \cdots & \gamma^{ff}(0) & \gamma^{fs}(0) & \cdots \\ \gamma^{sf}(1-p) & \gamma^{ss}(1-p) & \cdots & \gamma^{sf}(0) & \gamma^{ss}(0) & \cdots \end{array} \right] \quad (5.4) \\ & \text{Let } G = \mathbb{E} [x_t y_{t-1}^T] = \begin{bmatrix} \gamma^{ff}(0) & \gamma^{fs}(0) \\ \gamma^{sf}(0) & \gamma^{ss}(0) \\ \vdots & \vdots \\ \gamma^{ff}(1-p) & \gamma^{fs}(1-p) \\ \gamma^{sf}(1-p) & \gamma^{ss}(1-p) \end{bmatrix} = \Gamma \begin{bmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \text{ The block columns of the} \end{aligned}$$

second matrix on the right hand side of (5.4) are of the form

$$\begin{aligned} \mathbb{E} [x_t y_{t-j-1}^T] &= \mathbb{E} [Ax_{t-1} + B\varepsilon_{t-1}] y_{t-j-1}^T \\ &= \mathbb{E} [Ax_{t-1} y_{t-j-1}^T] = \mathcal{A}E [x_{t+j-1} y_{t-1}^T] = \cdots = \mathcal{A}^j G, \quad j \geq 0. \end{aligned} \quad (5.5)$$

Thus, this matrix can be written as $[G, AG, \mathcal{A}^2 G, \dots]$. From the Cayley Hamilton theorem, one can easily see that the second matrix on the right hand side of (5.4) has full-row rank if and only if the matrix consisting of the first np blocks has full-row rank. One can use only those equations in (5.4) where the columns on the left hand

side or the columns of the second matrix on the right hand side contain only second moments which can be observed. In other words one can consider the equation system

$$\mathbb{E} \left[y_t \begin{bmatrix} y_{t-1}^f, \dots, y_{t-np}^f \end{bmatrix}^T \right] = [A_1, \dots, A_p] \underbrace{\mathbb{E} \left[\begin{bmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{bmatrix} \begin{bmatrix} y_{t-1}^f \end{bmatrix}^T, \dots, \begin{bmatrix} y_{t-np}^f \end{bmatrix}^T \right]}_{=Z}. \quad (5.6)$$

Note that Chen and Zadrozny [1998] uses a larger subsystem of equations of (5.4) and thus, in particular, the identifiability result here, implies theirs.

As is easily seen, Z can be written as $[K, AK, A^2K, \dots, A^{np-1}K]$, where $K = \Gamma \begin{bmatrix} I_{n_f} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and therefore has the structure of a reachability matrix.

Clearly, the AR parameters A_1, \dots, A_p of the system (5.1) are identifiable if Z has full-row rank np , or equivalently the pair $[A, K]$ is reachable. It is shown in Anderson et al. [2012] that this rank condition is not necessary for identifiability.

5.3.2 Generic Identifiability of System Parameters

Consider the set of all AR systems for given order p and given rank q of the innovation covariance matrix satisfying the stability condition. As can easily be seen, the parameter space for this set is an open subset of $\mathbb{R}^{pn^2+nq-\frac{q(q-1)}{2}}$ if we take into account the uniqueness of the transfer function only up to orthogonal postmultiplication. A property on this parameter space is said to hold generically if it holds on a superset of an open and dense subset of the parameter space. For simplicity of notation, we do not take into account the restriction arising from making the orthogonal postmultiplication unique and thus consider a parameter space $\Theta \subseteq \mathbb{R}^{pn^2+nq}$. In this case, the notions of genericity are the same in both spaces, that means a generic set in Θ intersected with the zero restrictions corresponding to normalizing the orthogonal postmultiplication gives a generic set in the restricted parameter space.

The next theorem which is the central result of this chapter, show that the matrix Z in equation (5.6) is generically of full-row rank and thus all A_i matrices are generically identifiable. In addition, Σ is shown to be generically identifiable. Note that this holds both for regular and singular AR systems, for all sampling frequency ratios N , and all $n^f \geq 1$.

Theorem 5.3.1. *The matrix Z in equation (5.6) has full-row rank $n \cdot p$ on a generic subset of the parameter space Θ .*

The proof of the theorem, which uses system theoretic tools (see Hannan and Deistler [2012], Kailath [1980] and Anderson and Deistler [2008]), is quite intricate and we start by proving three lemmas and a corollary.

Lemma 5.3.2. Let \mathcal{A} denote the block companion matrix defined above, and let $\bar{A}(z)$ denote the polynomial matrix $z^p I - A_1 z^{p-1} - A_2 z^{p-2} - \dots - A_p$. Suppose that $\alpha^T = [\alpha_1^T, \alpha_2^T, \dots, \alpha_p^T]$, where $\alpha \in \mathbb{R}^{np}$, $\alpha_i \in \mathbb{R}^n$. If α^T is a left eigenvector of \mathcal{A} corresponding to eigenvalue λ , then $\alpha_1^T \neq 0$ and lies in the left kernel of $\bar{A}(\lambda)$, i.e. $\alpha_1^T \bar{A}(\lambda) = 0$. Conversely, if $\alpha_1 \neq 0$ is such that $\alpha_1^T \bar{A}(\lambda) = 0$, and $\alpha_2^T = \alpha_1^T(\lambda I - A_1)$, $\alpha_3^T = \alpha_1^T(\lambda^2 I - \lambda A_1 - A_2)$, \dots , holds for some λ , then $\alpha^T = [\alpha_1^T, \alpha_2^T, \dots, \alpha_p^T]$ is a left eigenvector of \mathcal{A} corresponding to eigenvalue λ .

Proof. Suppose that α^T is a left eigenvector of \mathcal{A} corresponding to eigenvalue λ . Then at once

$$\begin{aligned} \alpha_1^T A_1 + \alpha_2^T &= \lambda \alpha_1^T \\ \alpha_1^T A_2 + \alpha_3^T &= \lambda \alpha_2^T \\ &\vdots \\ \alpha_1^T A_{p-1} + \alpha_p^T &= \lambda \alpha_{p-1}^T \\ \alpha_1^T A_p &= \lambda \alpha_p^T \end{aligned} \tag{5.7}$$

Evidently, $\alpha_1 \neq 0$, else these equations yield that α itself would be zero, contradicting the fact that α is an eigenvector. It is trivial to eliminate $\alpha_2, \alpha_3, \dots$ to obtain

$$\alpha_1^T (\lambda^p I - \lambda^{p-1} A_1 - \lambda^{p-2} A_2 - \dots - A_p) = 0 \tag{5.8}$$

Conversely, suppose $\alpha_1^T \bar{A}(\lambda) = 0$ with $\alpha_1 \neq 0$, and that $\alpha_2, \alpha_3 \dots$ are defined as in the lemma hypothesis. Equations (5.7) easily follow and then the eigenvalue property $\alpha^T \mathcal{A} = \lambda \alpha^T$ is immediate. \square

Corollary 5.3.3. With \mathcal{A}, \mathcal{B} as above, the pair $[\mathcal{A}, \mathcal{B}]$ is reachable for a generic subset of the parameter space.

Proof. We argue that generically, for each value of λ for which $\bar{A}(\lambda)$ is singular, the left kernel is one-dimensional.

Let $\alpha_1 \neq 0$ be such that $\alpha_1^T \bar{A}(\lambda) = 0$. Then we can find a nonsingular matrix T such that the first row of $T \bar{A}(\lambda)$ is zero. Thus,

$$0 = \alpha_1^T T^{-1} T \bar{A}(\lambda)$$

and thus the first element of $[\alpha_1^T T^{-1}]$ can be chosen to be nonzero. Now, consider the polynomial matrix formed by the rows 2 till n of the polynomial matrix $T \bar{A}(z)$. By the result on generic zerolessness of tall rational transfer functions, see Anderson and Deistler [2008], the components 2 till n of the vector $[\alpha_1^T T^{-1}]$ must be zero which gives the desired result. For any nonzero vector α , say, generating such a left kernel, there holds $\alpha^T \mathcal{B} = \alpha_1^T b \neq 0$, by virtue of the genericity, and this will hold for an arbitrary but finite number of α . Now the inequality above is easily seen to hold in an open subset of the parameter space: This is a consequence of the continuity of the mapping attaching a suitably normalized eigenvector α of \mathcal{A} to \mathcal{A} and the continuity

of the inner product $\alpha^T B$. This means that reachability is established in an open subset of the parameter space. The density of this set is shown as follows: If $\alpha_1^T b = 0$, then there is a sequence $b_i \rightarrow b$ such that $\alpha_1^T b_i \neq 0$. \square

Lemma 5.3.4. *Let \mathcal{A} denote the block companion matrix defined above, and let $\bar{A}(z)$ denote the polynomial matrix defined above. Suppose that c is an np -vector, with $c = [c_1^T, c_2^T, \dots, c_p^T]^T$ and each c_i an n -vector. Then if c is a right eigenvector of \mathcal{A} corresponding to eigenvalue $\lambda \neq 0$, then $c_1 \neq 0$ and lies in the kernel of $\bar{A}(\lambda)$. Conversely, if $c_1 \neq 0$ is such that $\bar{A}(\lambda)c_1 = 0$ with $\lambda \neq 0$ and $c_i = \lambda^{-i+1}c_1$, then $c = [c_1^T, c_2^T, \dots, c_p^T]^T$ is a right eigenvector of \mathcal{A} with eigenvalue λ .*

Proof. Assume that $\mathcal{A}c = \lambda c$. Then it is easily seen that

$$\begin{aligned} \sum_{i=1}^p A_i c_i &= \lambda c_1 \\ c_1 &= \lambda c_2 \\ c_2 &= \lambda c_3 \\ &\vdots \\ c_{p-1} &= \lambda c_p \end{aligned}$$

The result is easily proved using these equations. \square

Denote by e_j the n -vector with 1 in the j -th entry and all other entries zero. Denote by \mathcal{E}_j the np -vector $\mathcal{E}_j = [e_j^T, 0, 0, \dots, 0]^T$. Then we have

Lemma 5.3.5. *The pair $[\mathcal{A}, \mathcal{E}_j]$ is observable on a generic subset of the parameter space Θ .*

Proof. It is well known that nonobservability of the pair $[\mathcal{A}, \mathcal{E}_j]$ is equivalent to the existence of a nonzero vector c for which, for some λ , there holds $\mathcal{A}c = \lambda c$, $\mathcal{E}_j^T c = 0$.

Suppose the conclusion of the lemma is false. Let c_1 denote the vector comprising the first n entries of c and let λ be the corresponding eigenvalue of \mathcal{A} which is nonzero because A_p can be assumed as nonsingular on a generic set. Then by the result of Lemma 5.3.4, it holds that

$$\begin{aligned} \bar{A}(\lambda)c_1 &= 0 \\ e_j^T c_1 &= 0. \end{aligned}$$

Now observe that the second equation requires that the j -th entry of c_1 be zero. From the generic zerolessness of tall transfer functions (see Anderson and Deistler [2008]), we conclude that the other $n - 1$ entries of c_1 must be zero for a generic subset of the parameter space, i.e. that c_1 itself is zero. This is clearly false. Hence the conclusion of the lemma is established by contradiction. \square

With the help of Corollary 5.3.3 and Lemma 5.3.5 and 5.3.4, we are now able to provide proof for the main theorem of this subsection i.e. Theorem 5.3.1.

Proof of Theorem 5.3.1

Proof. The following equations can be verified using equation (5.3)

$$(zI - \mathcal{A})\Gamma(z^{-1}I - \mathcal{A}^T) + \mathcal{A}\Gamma(z^{-1}I - \mathcal{A}^T) + (zI - \mathcal{A})\Gamma\mathcal{A}^T = \mathcal{B}\mathcal{B}^T \quad (5.9)$$

$$\Gamma + (zI - \mathcal{A})^{-1}\mathcal{A}\Gamma + \Gamma\mathcal{A}^T(z^{-1}I - \mathcal{A}^T)^{-1} = (zI - \mathcal{A})^{-1}\mathcal{B}\mathcal{B}^T(z^{-1}I - \mathcal{A}^T)^{-1}. \quad (5.10)$$

Premultiplying and postmultiplying by \mathcal{E}_1^T and \mathcal{E}_1 leads to

$$\begin{aligned} \mathcal{E}_1^T\Gamma\mathcal{E}_1 + \mathcal{E}_1^T(zI - \mathcal{A})^{-1}\mathcal{A}\Gamma\mathcal{E}_1 + \mathcal{E}_1^T\Gamma\mathcal{A}^T(z^{-1}I - \mathcal{A}^T)^{-1}\mathcal{E}_1 = \\ \mathcal{E}_1^T(zI - \mathcal{A})^{-1}\mathcal{B}\mathcal{B}^T(z^{-1}I - \mathcal{A}^T)^{-1}\mathcal{E}_1. \end{aligned} \quad (5.11)$$

Since by corollary 5.3.3 $[\mathcal{A}, \mathcal{B}]$ is reachable on a generic subset of the parameter space, and by lemma 5.3.5, we have that $[\mathcal{A}, \mathcal{E}_1]$ is observable on a generic subset of the parameter space, and since the intersection of two generic sets is generic again, $[\mathcal{A}, \mathcal{B}, \mathcal{E}_1]$ is minimal and thus the McMillan degree of $\mathcal{E}_1^T(zI - \mathcal{A})^{-1}\mathcal{B}$ is np , the dimension of \mathcal{A} . It follows that the transfer function $\mathcal{B}^T(z^{-1}I - \mathcal{A}^T)^{-1}\mathcal{E}_1$ also has McMillan degree np , since transpose operations preserve McMillan degree, and replacements of a variable by a Mobius transformation (see Kalman [1965]) of that variable preserve McMillan degree.

Further, by the stability assumption on the underlying AR system and because A_p is assumed as nonsingular, the McMillan degree of the product $\mathcal{E}_1^T(zI - \mathcal{A})^{-1}\mathcal{B}\mathcal{B}^T(z^{-1}I - \mathcal{A}^T)^{-1}\mathcal{E}_1$ will be $2np$, due to the absence of any pole-zero cancellations. Now the two nonconstant transfer functions in (5.11) on the left side necessarily have the same McMillan degree, one being obtainable from the other by transposition and Mobius transformation of the variable. Further, the nonconstant transfer functions on the left side of equation share no common poles, so that their sum has McMillan degree equal to the sum of the two McMillan degrees, or twice the McMillan degree of one of the transfer functions. Hence on the left side, we must have the McMillan degree of $\mathcal{E}_1^T(zI - \mathcal{A})^{-1}\mathcal{A}\Gamma\mathcal{E}_1$ equal to np , so that $\mathcal{A}, \mathcal{A}\Gamma\mathcal{E}_1$ is reachable. It also follows that $[\mathcal{A}, \Gamma\mathcal{E}_1]$ is reachable.

Up to now, the proof has only been given for the case $n_f = 1$. As is easily seen, the result is true a fortiori for $n_f > 1$. \square

The importance of theorem 5.3.1 is that if Z has full-row rank, the mapping from the second moments which can be observed in principle to the parameters is continuous and thus consistent estimators of the corresponding population second moments give consistent estimators for the underlying high frequency parameters. This also holds for the case of generalized factor models where the static factors can be estimated by principal component analysis, see Deistler et al. [2010b].

5.3.3 Generic Identifiability of the Noise Parameters

To show generic identifiability of the noise parameters Σ , one can commence from identifiable system parameters A_1, \dots, A_p . Rewriting equation (5.2) as

$$\underbrace{\begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix}}_{=x_{t+1}} = \mathcal{A} \underbrace{\begin{bmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{bmatrix}}_{=x_t} + \underbrace{\begin{bmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=\mathcal{G}} v_t. \quad (5.12)$$

$$y_t = \underbrace{[I_n, 0, \dots, 0]}_{=\mathcal{H}} x_{t+1} \quad (5.13)$$

one can conclude through vectorization of

$$\Gamma = \mathbb{E} [x_t x_t^T] = \mathcal{A} \Gamma \mathcal{A} + \mathcal{G} \Sigma \mathcal{G}$$

and

$$\gamma(0) = \mathbb{E} [y_t y_t^T] = \mathcal{H} \Gamma \mathcal{H}^T$$

that

$$\begin{aligned} \text{vec}(\Gamma) &= (\mathcal{A} \otimes \mathcal{A}) \text{vec}(\Gamma) + (\mathcal{G} \otimes \mathcal{G}) \text{vec}(\Sigma) \\ &= (I - \mathcal{A} \otimes \mathcal{A})^{-1} (\mathcal{G} \otimes \mathcal{G}) \text{vec}(\Sigma) \end{aligned} \quad (5.14)$$

and

$$\text{vec}(\gamma(0)) = (\mathcal{H} \otimes \mathcal{H}) \text{vec}(\Gamma). \quad (5.15)$$

Therefore, we obtain that

$$\text{vec}(\gamma(0)) = (\mathcal{H} \otimes \mathcal{H}) (I - \mathcal{A} \otimes \mathcal{A})^{-1} (\mathcal{G} \otimes \mathcal{G}) \text{vec}(\Sigma). \quad (5.16)$$

Note that the absolute value of all eigenvalues of \mathcal{A} is smaller than one by the stability assumption and therefore the same holds for the eigenvalues of $(\mathcal{A} \otimes \mathcal{A})$. This implies that $(I - \mathcal{A} \otimes \mathcal{A})$ is regular. For $A_1 = \dots = A_p = 0$, the matrix $(I - \mathcal{A} \otimes \mathcal{A})^{-1}$ is triangular with ones on its diagonal. Thus $(\mathcal{H} \otimes \mathcal{H})(I - \mathcal{A} \otimes \mathcal{A})^{-1}(\mathcal{G} \otimes \mathcal{G})$ is a principal submatrix with the same property and is therefore nonsingular. This nonsingularity holds in an open neighborhood of $A_1 = \dots = A_p = 0$ and this neighborhood has a nonempty intersection with the generic set of identifiable system parameters as described in theorem 1. Now, there exists a point in this intersection for which the determinant of $(\mathcal{H} \otimes \mathcal{H})(I - \mathcal{A} \otimes \mathcal{A})^{-1}(\mathcal{G} \otimes \mathcal{G})$ is unequal to zero. Since this determinant is a rational function in the free entries A_1, \dots, A_p the nonsingularity holds for a generic set in the parameter space. For the properties of the set of zeros of multivariate polynomials and thus rational functions (see Shafarevich [1994]).

Clearly, if the matrix on the right hand side of equation (5.16) is nonsingular

the identifiability of Σ is achieved. Hence, using the above argument, the following desirable result is attained.

Theorem 5.3.6. *The noise parameters Σ are generically identifiable.*

Note that, as immediate from the proof above, for generic values of A_1, \dots, A_p, Σ is always identifiable.

We now introduce a numerical example to further explain the main results of this chapter.

Example 5.3.7. *Let us consider the following regular AR system*

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + v_t. \quad (5.17)$$

We suppose that $n = 2$, $n_f = 1$ and $n_s = 1$ and v_t is a zero mean white noise signal with an unknown covariance. Then for a generic choice of matrices A_1 and A_2 , one can easily verify that the corresponding Z matrix has rank equal to four i.e. it has full-row rank. However, when the parameter matrices A_1 and A_2 are both assigned to a nongeneric value of $\begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$ the Z matrix has rank two. This means that the matrices A_1 , A_2 and covariance of the input noise cannot be obtained through our proposed method.

5.4 Summary

This chapter mainly focused on the identifiability of AR systems when some parts of the observation vector are alternately missing. The method of modified extended Yule-Walker equations was proposed. Then this technique was used to suggest an algorithm whereby a mixed frequency AR system might be identified. Implementation of the algorithm depends on a certain condition, the satisfaction of which was established for generic mixed frequency AR systems.

On the Identifiability of Singular Autoregressive Models from Mixed Frequency Data - Linearly Dependent Lags

Abstract

This chapter is focused on the identifiability of an underlying high frequency multivariate stable singular AR system from mixed frequency data. In particular, this chapter is concerned with stable singular AR systems where the covariance matrix associated with the vector obtained by stacking observation vector, y_t , and its lags from the first lag to the p -th one (p is the order of the AR system), is also singular. To handle this scenario, it is assumed that the column degrees of the associated polynomial matrix are known. It is clearly demonstrated that the system and noise parameters are all generically identifiable.

6.1 Introduction

With recent theoretical advances in the field of econometric modeling, see Forni et al. [2000] and Deistler et al. [2010b], singular autoregressive models have become more popular in this area. In econometric modeling and forecasting exercises using generalized dynamic factor models (GDFMs) Banbura et al. [2010], the latent variable, i.e. those parts of observed data remaining after removal of contaminating additive noise in the measurement, are modeled as a singular autoregressive model. In this chapter, the identifiability of this type of model is studied in full detail.

This chapter is in fact a continuation of the work in the previous chapter. The last chapter showed the identifiability only on a subset of the parameter space where Γ , the covariance matrix associated with the vector $Y_t = \begin{bmatrix} y_t^T & y_{t-1}^T & \cdots & y_{t-p}^T \end{bmatrix}^T$, where y_t is the observation vector and p is the order of the AR system, is restricted to being nonsingular. Note that Γ is always nonsingular for regular AR models

but can be singular for singular AR models. In this chapter, we explore generic identifiability for singular autoregressive models where the covariance matrix Γ is singular. Following the reference Deistler et al. [2011], it is assumed that the column degrees of the associated polynomial matrix are known. Later in this chapter we provide more details about this. Similar to the previous chapter, the extended Yule-Walker equations proposed in Chen and Zadrozny [1998] are modified to obtain a sufficient condition for identifiability and then again to show that this condition generically holds.

The rest of this chapter is organized as follows. First, the problem formulation is provided. Then we focus on possible scenarios and study them separately. Considering AR polynomials with prescribed column degrees, we distinguish two cases: the case where the AR polynomial matrix $A(q)$ has columns with unequal degree but there is no column with zero column degree and another case where columns of $A(q)$ are permitted to have column degree zero. The former case is studied in Section 6.3 and the latter is explored in Section 6.4. Finally, Section 6.5 provides concluding remarks.

6.2 Problem Formulation

Consider the following AR system

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + A_3 y_{t-3} + \dots + A_p y_{t-p} + v_t, \quad (6.1)$$

where y_t is an \mathbb{R}^n valued random variable. Here, it is assumed that y_t consists of two parts; the fast components, y_t^f for $t \in \mathbb{Z}$, which are n_f -dimensional, and the slow components, y_t^s which are available for $t \in N\mathbb{Z}$ for some positive integer $N \geq 2$, which are n_s -dimensional, and $n_f + n_s = n$. The innovation v_t , which is orthogonal to $y_{t-j}, j \geq 1$ is white noise and its covariance is $\mathbb{E}[v_t v_u^T] = \Sigma \delta_{tu}$ for all t, u , where δ_{tu} is the Kronecker delta, which is 1 for $t = u$ and 0 otherwise, $\text{rank}(\Sigma) = r < n$. Note that the case $\text{rank}(\Sigma) = n$ implies that Γ is nonsingular and thus this case is already treated completely in the previous chapter. One can write $\Sigma = bb^T$ where b is an $n \times r$ matrix. Accordingly, $v_t = b\varepsilon_t$, where $\mathbb{E}[\varepsilon_t \varepsilon_t^T] = I_r$. For given Σ , b is unique up to postmultiplication by an orthogonal matrix. Moreover, the AR polynomial matrix is $A(q) = I - A_1 q - \dots - A_p q^p$ where $qy_t = y_{t-1}$ and $z = q^{-1}$, so q is the backward shift. We also define the polynomial matrix

$$\bar{A}(z) \triangleq z^p A(q) = z^p A(z^{-1}) \quad (6.2)$$

that is needed later in this chapter. Throughout this chapter, it is assumed that the high frequency system (6.1) is stable, and that we restrict ourselves to the steady state and thus stationary solutions.

Now it is convenient to define the state variable x_t as below and rewrite the

equation (6.1) in the state space form as

$$\underbrace{\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} A_1 & A_2 & \dots & A_p \\ I & 0 & \dots & 0 \\ & \ddots & \ddots & \\ 0 & 0 & I & 0 \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix} b \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_B \epsilon_t. \quad (6.3)$$

The previous chapter studied identifiability, i.e. whether the system parameters $[A_1, \dots, A_p]$, and noise parameters Σ can be uniquely determined from those population second moments which can be observed in principle, these being $\gamma^{ff}(h) = \mathbb{E} \left[y_{t+h}^f [y_t^f]^T \right]$, $h \in \mathbb{Z}$; $\gamma^{fs}(h) = \mathbb{E} \left[y_{t+h}^f [y_t^s]^T \right]$, $h \in \mathbb{Z}$; $\gamma^{ss}(h) = \mathbb{E} \left[y_{t+h}^s [y_t^s]^T \right]$, $h \in \mathbb{N}\mathbb{Z}$. Moreover, identifiability of the system and noise parameters was shown on a generic set. This generic set was the set of all stable AR systems where A_p was nonsingular and the eigenvalues of \mathcal{A} had multiplicity one. The assumption that A_p is nonsingular implies the property that all column degrees in $A(q)$ are equal to p , which is restrictive. However, here we study the scenario where at least one column degree is less than p and thus A_p is singular, though nonzero. Here, the assumption is that prescribed column degrees of $A(q)$ are known. The parameters in the coefficient matrices of $A(q)$ not forced to be zero by the column degree restriction together with the $nr - \frac{r(r-1)}{2}$ free entries of Σ , or b , then define our parameter space.

In the following, two possible scenarios which may happen are considered. First, we study the case where $A(q)$ has columns with unequal degree but there is no column with column degree zero. Second, we explore a situation where columns of $A(q)$ are permitted to have column degree zero. In the following section, the former case is examined and in Section 6.4 the latter case is investigated.

6.3 AR Systems with Unequal Column Degree in $A(q)$

Throughout the chapter, we assume for convenience and without loss of essential generality that the components of y_t are ordered such that the column degrees of $A(q)$, the AR polynomial, are decreasing, i.e., $p_i \geq p_j$ when $i < j$ where i and j denotes a column numbers in $A(q)$ and p_i and p_j are the corresponding column degrees. Also, it is admitted that $p_j > 0$.

Now, to deal with the situation stated above the following state space form is studied

$$\bar{x}_{t+1} = \bar{\mathcal{A}}\bar{x}_t + \bar{B}\epsilon_t, \quad (6.4)$$

where $\bar{\mathcal{A}}$ is obtained from \mathcal{A} by deleting columns corresponding to prescribed zero columns in A_1, \dots, A_p and corresponding rows. This has been called the quasi companion form in Deistler et al. [2011]. Here, \bar{x}_{t+1} and \bar{B} are those entries of x_{t+1} and

B respectively associated with \bar{A} . Furthermore, it is easy to verify that

$$y_t = [\bar{A}_1 \bar{A}_2 \dots \bar{A}_p] \bar{x}_t + b e_t. \quad (6.5)$$

where $[\bar{A}_1 \bar{A}_2 \dots \bar{A}_p]$ is obtained by taking the first n rows of \bar{A} . Note that $\bar{A}_1 = A_1$ since no column degree is prescribed to be zero; however, \bar{A}_i , $i \in \{2, \dots, p\}$, may have fewer columns than A_i . Since only prescribed zero columns are deleted, identifiability of the system parameters is equivalent to identifiability of $[\bar{A}_1 \bar{A}_2 \dots \bar{A}_p]$. In the following, we first demonstrate that the parameter matrices \bar{A}_i and Σ are generically identifiable from those population second moments, which can be observed in principle.

6.3.1 Modified Extended Yule-Walker Equations

Consider the system (6.5); then it is easy to see that

$$\mathbb{E}[y_t \bar{x}_t^T] = [\bar{A}_1 \bar{A}_2 \dots \bar{A}_p] \bar{\Gamma}, \quad (6.6)$$

where $\bar{\Gamma} = \mathbb{E}[\bar{x}_t \bar{x}_t^T]$. Observe that provided $\bar{\Gamma}$ and $\mathbb{E}[y_t \bar{x}_t^T]$ are known and $\bar{\Gamma}$ is non-singular, one can identify the parameters \bar{A}_i easily using (6.6). However, one has difficulties in directly using (6.6) because y_t^s is not available at all times and consequently some entries of the matrices on both sides of (6.6) will be missing. In the rest of this subsection, it is first illustrated how those population second moments, which can be observed in principle, can be used to determine the \bar{A}_i . Then, later these results are used for proving generic identifiability of Σ .

To overcome the problem of missing covariance data, one may start from equation (6.5) and postmultiply both sides by $[y_{t-j}^f]^T$, $j > 0$, the fast components, to obtain the modified extended Yule-Walker equations as in Anderson et al. [2012].

$$\mathbb{E} \left[y_t \left[(y_{t-1}^f)^T, (y_{t-2}^f)^T, \dots \right] \right] = [\bar{A}_1, \dots, \bar{A}_p] \mathbb{E} \left[\bar{x}_t \left[(y_{t-1}^f)^T, (y_{t-2}^f)^T, \dots \right] \right]. \quad (6.7)$$

From equation (6.4) one can attain for the second multiplicand on the right hand side

$$\begin{aligned} \mathbb{E} \left[\bar{x}_{t+1} y_{t-j}^T \right] &= \mathbb{E} \left[[\bar{A} \bar{x}_t + \bar{B} e_t] y_{t-j}^T \right] = \\ \bar{A} \mathbb{E} \left[\bar{x}_t y_{t-j}^T \right] &= \dots = \bar{A}^j \mathbb{E} \left[\bar{x}_{t-j-1} y_{t-j}^T \right] = \bar{A}^j \bar{\Gamma} \begin{bmatrix} I_n \\ 0 \end{bmatrix}. \end{aligned} \quad (6.8)$$

We now define $K = \bar{\Gamma} \begin{bmatrix} I_{n_f} \\ 0 \end{bmatrix}$.

The rightmost matrix in (6.7) can be written as $[K, \bar{A}K, \bar{A}^2K, \dots]$. Now as $\bar{A} \in \mathbb{R}^{(pn-s) \times (pn-s)}$, where s is the number of prescribed zero columns in $[A_1, \dots, A_p]$, using the Cayley-Hamilton theorem, it follows that this matrix has full-row rank if

and only if the following matrix which contains only the first $pn - s$ block columns has full-row rank:

$$\mathcal{Z}^f = [K, \bar{\mathcal{A}}K, \bar{\mathcal{A}}^2K, \dots, \bar{\mathcal{A}}^{pn-s-1}K]. \quad (6.9)$$

Now obviously the parameter matrices \bar{A}_i are identifiable if the matrix \mathcal{Z}^f has full-row rank.

6.3.2 Generic Identifiability

Here, the same definition of generic identifiability as in Chapter 5 is followed. Consider the parameter space associated with the system (6.5). Then a property is said to hold generically on the parameter space if it holds on a superset of an open and dense subset of the parameter space. In what follows first, the generic identifiability of the system parameters $[\bar{A}_1, \dots, \bar{A}_p]$ from those second moments which are observed in principle is analyzed. Then later in Subsection 6.3.2.2, the generic identifiability of noise parameters i.e. the entries of Σ is examined.

6.3.2.1 Generic Identifiability of the System Parameters

In this subsection, it is shown that \mathcal{Z}^f has generically full-row rank and thus we have generic identifiability of the system parameters $[\bar{A}_1 \bar{A}_2 \dots \bar{A}_p]$ for prescribed column degrees.

Lemma 6.3.1. *Let $\bar{\mathcal{A}}$ denote the block matrix defined in (6.4) and let $\bar{A}(z)$ denote the polynomial matrix defined in (6.2). Suppose that $\bar{\alpha}^T = [\bar{\alpha}_1^T \dots \bar{\alpha}_p^T]$ where $\bar{\alpha}_i$ has dimension equal to the number of columns of \bar{A}_i , is a left eigenvector of $\bar{\mathcal{A}}$ corresponding to eigenvalue λ . Then $\bar{\alpha}_1^T$ is in the left kernel of $\bar{A}(\lambda)$ i.e. $\bar{\alpha}_1^T \bar{A}(\lambda) = 0$. Conversely, if $\bar{\alpha}_1^T \neq 0$ is such that $\bar{\alpha}_1^T \bar{A}(\lambda) = 0$ for some λ , and $[\bar{\alpha}_2^T 0 \dots 0] = \bar{\alpha}_1^T (\lambda I - A_1)$, $[\bar{\alpha}_3^T 0 \dots 0] = \bar{\alpha}_1^T (\lambda^2 I - \lambda A_1 - A_2)$, ..., then $\bar{\alpha}^T = [\bar{\alpha}_1^T \bar{\alpha}_2^T \bar{\alpha}_3^T \dots \bar{\alpha}_p^T]$ is a left eigenvector of $\bar{\mathcal{A}}$ corresponding to eigenvalue λ ; here, the number of zero entries in α_i , $i = 2, \dots, p$, is equal to the number of prescribed zero columns in A_i .*

Proof. Suppose that $\bar{\alpha}^T$ is a left eigenvector of $\bar{\mathcal{A}}$ associated with eigenvalue λ . Then it is evident that

$$\begin{aligned} \bar{\alpha}_1^T \bar{A}_1 + [\bar{\alpha}_2^T 0 \dots 0] &= \lambda \bar{\alpha}_1^T, \\ \bar{\alpha}_1^T \bar{A}_2 + [\bar{\alpha}_3^T 0 \dots 0] &= \lambda \bar{\alpha}_2^T, \\ &\vdots \\ \bar{\alpha}_1^T \bar{A}_{p-1} + [\bar{\alpha}_p^T 0 \dots 0] &= \lambda \bar{\alpha}_{p-1}^T, \\ \bar{\alpha}_1^T \bar{A}_p &= \lambda \bar{\alpha}_p^T. \end{aligned} \quad (6.10)$$

In the above equations, in the second summand on the left hand side, the $\bar{\alpha}_i$ are augmented with zeros so the above equalities can hold with dimensional consistency. Now if $\bar{\alpha}_1$ were zero then all $\bar{\alpha}_i$ would turn out to be zero which would be a contradiction. Thus, $\bar{\alpha}_1 \neq 0$. Now, let α_i denote the vector consisting of $\bar{\alpha}_i$ augmented with zeros if necessary to make its dimension equal to $\dim y_t$. Note that we necessarily have $\bar{\alpha}_1 = \alpha_1$. Then it is obvious that the following equations hold

$$\begin{aligned}\bar{\alpha}_1^T A_1 + \alpha_2^T &= \lambda \bar{\alpha}_1^T, \\ \bar{\alpha}_1^T A_2 + \alpha_3^T &= \lambda \alpha_2^T, \\ &\vdots \\ \bar{\alpha}_1^T A_{p-1} + \alpha_p^T &= \lambda \alpha_{p-1}^T, \\ \bar{\alpha}_1^T A_p &= \lambda \alpha_p^T.\end{aligned}\tag{6.11}$$

Thus, it is easy to obtain $\bar{\alpha}_1 \bar{A}(\lambda) = 0$. Conversely, suppose that $\bar{\alpha}_1 \bar{A}(\lambda) = 0$ holds for some nonzero λ . Then by defining $\bar{\alpha}_i$ according to the lemma statement, it can be easily verified that (6.10) holds and thus $\bar{\alpha}^T \bar{A} = \lambda \bar{\alpha}^T$. \square

Theorem 6.3.2. *The set $\mathcal{F} = \{[\bar{A}, \bar{B}] | \text{rank}([zI - \bar{A}, \bar{B}]) = pn - s, \forall z \in \mathbb{C}\}$, where s is the number of prescribed zero columns in (A_1, \dots, A_p) , is open and dense in the set of all \bar{A}, \bar{B} satisfying the conditions described above and where \bar{A} corresponds to a stable $A(q)$.*

Proof. Dense: Consider the pair $[\bar{A}_0, \bar{B}_0]$, where \bar{A}_0 and \bar{B}_0 are a particular choice of \bar{A} and \bar{B} , which is not reachable. Note that \bar{A}_0 has a finite set of eigenvalues and there exists a finite set of vectors $\bar{\alpha}_{1i}^T$ $i = 1, 2, \dots, \dim \bar{A}_0$ each defined to within a nonzero scaling constant and comprising the first n entries of the associated left eigenvectors. Now without the loss of generality, suppose that \bar{B}_0 is a column vector (which would be the worst case scenario) and let \bar{b}_0 denote its first n components. Now using the Popov-Belevitch-Hautus test (see Kailath [1980]), the pair \bar{A}_0 and \bar{B}_0 is unreachable if and only if there exist some i such that $\bar{\alpha}_{1i}^T \bar{b}_0 = 0$. Note that the orthogonal complement of $\bar{\alpha}_{1i}^T$ is $n - 1$ dimensional. So, we can easily find a sequence $\bar{b}_{0j} \rightarrow \bar{b}_0$ such that $\bar{\alpha}_{1i}^T \bar{b}_{0j} \neq 0$.

Open: Set \mathcal{F} is open if and only if \mathcal{F}^C (the complement of \mathcal{F}) is closed. To obtain a contradiction assume that \mathcal{F}^C is not closed. Now, consider pairs $[\bar{A}_m, \bar{B}_m] \in \mathcal{F}^C$, where $m \in \mathbb{N}$, and $[\bar{A}_0, \bar{B}_0] \in \mathcal{F}$ with $\bar{A}_m \rightarrow \bar{A}_0$ and $\bar{B}_m \rightarrow \bar{B}_0$. Since $[\bar{A}_m, \bar{B}_m] \in \mathcal{F}^C$ then there exists an eigenvalue of \bar{A}_m , say z_m , such that $\text{rank}([z_m I - \bar{A}_m, \bar{B}_m]) < pn - s$. Without significant loss of generality, we assume that $\text{rank}([z_m I - \bar{A}_m, \bar{B}_m])$ is strictly one less the number of rows. Furthermore, associated with z_m there exists an eigenvector of \bar{A}_m with a unit length, r_m^T , which must be orthogonal to the columns of \bar{B}_m . Now consider pair $[\bar{A}_0, \bar{B}_0]$, based on the assumption that $[\bar{A}_0, \bar{B}_0] \in \mathcal{F}$, the augmented matrix $[zI - \bar{A}_0, \bar{B}_0]$ is of full-row rank for all eigenvalues of \bar{A}_0 . Consider one of the eigenvalues of \bar{A}_0 which is limit of z_m , say z_0 , and let r_0^T represent the associated eigenvector. Then according to the same assumption r_0^T is not orthogonal to the columns of \bar{B}_0 . Now as $\bar{A}_m \rightarrow \bar{A}_0$, it is easy to see that $z_m \rightarrow z_0$; moreover,

$r_m^T \rightarrow r_0^T$ also $\bar{B}_m \rightarrow \bar{B}_0$. Additionally, we know that r_m^T is orthogonal to the columns of \bar{B}_m but r_0^T is not orthogonal to the columns of \bar{B}_0 . This results in a contradiction in light of the fact that $r_m^T \bar{B}_m \rightarrow r_0^T \bar{B}_0$.

□

Lemma 6.3.3. *Let \bar{A} denote the block matrix defined in (6.4) and let $\bar{A}(z)$ denote the polynomial matrix defined in (6.2). Suppose that \bar{c} is a right eigenvector of \bar{A} corresponding to eigenvalue $\lambda \neq 0$. Partition $\bar{c} = [\bar{c}_1^T \bar{c}_2^T \dots \bar{c}_p^T]^T$ where \bar{c}_i has the same number of entries as columns of \bar{A}_i . Then $\bar{c}_1 \neq 0$ is in the kernel of $\bar{A}(\lambda)$.*

Conversely, if $\bar{c}_1 \neq 0$ is such that $\bar{A}(z)\bar{c}_1 = 0$ for some $\lambda \neq 0$ and if $c_i = \lambda^{1-i}\bar{c}_1$ and \bar{c}_i denotes the first n_i entries of c_i , where n_i is the number of columns in \bar{A}_i , then $\bar{c} = [\bar{c}_1^T \bar{c}_2^T \dots \bar{c}_p^T]^T$ is the right eigenvector of \bar{A} corresponding to eigenvalue $\lambda \neq 0$.

Proof. First note that there exists a permutation matrix P and a matrix U and lower triangular matrix N , the latter matrices have entries of 1 or 0, such that

$$PAP^T = \begin{bmatrix} \bar{A} & 0 \\ U & N \end{bmatrix}. \quad (6.12)$$

Suppose that \bar{c} is an eigenvector for \bar{A} corresponding to eigenvalue $\lambda \neq 0$ then the following vector is an eigenvector of PAP^T corresponding to the same eigenvalue

$$\begin{bmatrix} \bar{c} \\ (\lambda I - N)^{-1}U\bar{c} \end{bmatrix}. \quad (6.13)$$

Thus, there is an eigenvector of A corresponding to eigenvalue λ of the form $c = [c_1^T c_2^T \dots c_p^T]^T$ where the first entries of c_i comprise of \bar{c}_i and the remainder are obtained from $(\lambda I - N)^{-1}U\bar{c}$. Moreover, due to the structure of A , we can conclude that $c_1 = \bar{c}_1$ and $c_i = \lambda^{-i+1}\bar{c}_1$ and then that $\bar{A}(\lambda)\bar{c}_1 = 0$.

For the converse part, suppose that $\bar{A}(\lambda)\bar{c}_1 = 0$ with $\lambda \neq 0$. It can be easily verified that with $c_i = \lambda^{-i+1}\bar{c}_1$, the vector $c = [c_1^T c_2^T \dots c_p^T]^T$ is an eigenvector of A corresponding to eigenvalue λ . Now by using the equations (6.12) and (6.13), we can conclude that any nonzero eigenvalue of PAP^T is also a nonzero eigenvalue of \bar{A} with an eigenvector of the latter matrix determined as a subvector of the former matrix.

□

Theorem 6.3.4. *Let \bar{A} denote the block matrix defined in (6.4). Furthermore, let \mathcal{E}_j denote a column vector of length equal to $\dim \bar{A}$ with one in the j -th position for $1 \leq j \leq \dim y_t$ and zero elsewhere. Then the pair $[\bar{A}, \mathcal{E}_j^T]$ is observable on a generic subset of the parameter space.*

Proof. It is well known that the pair $[\bar{A}, \mathcal{E}_j^T]$ is unobservable if and only if there exists a λ and a nonzero vector \bar{c} such that $\bar{A}\bar{c} = \lambda\bar{c}$ and $\mathcal{E}_j^T\bar{c} = 0$. Now, suppose that the conclusion of the lemma is false. Let \bar{c}_1 denote the first n entries of \bar{c} and λ be the eigenvalue associated with the eigenvector \bar{c} . Then λ is nonzero because \bar{A} can be

assumed nonsingular on a generic set. Using the result of Lemma 6.3.3, the following equations hold

$$\bar{A}(\lambda)\bar{c}_1 = 0 \text{ and } e_j^T \bar{c}_1 = 0$$

where e_j has dimension equal to that of y_t and is a unit vector with 1 in the j -th position. The second equality requires the j -th entry of \bar{c}_1 to be zero. Using the fact that tall transfer functions are generically zero-free (see Anderson et al. [2013a]), we can conclude that the rest of the entries of \bar{c}_1 must also be zero, so \bar{c}_1 is itself zero. The latter is obviously false in the light of Lemma 6.3.3. Thus, the conclusion of the theorem readily follows. \square

Proof of the next theorem is similar to the proof of Theorem 5.3.1 in the previous chapter. Here, we slightly modify that proof inline with contents of this chapter.

Theorem 6.3.5. *The matrix Z^f has full-row rank for a set of generic parameter matrices \bar{A}_i , $i = 1, 2, \dots, p$.*

Proof. Consider the system (6.4). Then observe that the following equality holds:

$$\bar{\Gamma} - \bar{\mathcal{A}}\bar{\Gamma}\bar{\mathcal{A}}^T = \bar{\mathcal{B}}\bar{\mathcal{B}}^T. \quad (6.14)$$

From (6.14) one can easily write

$$(zI - \bar{\mathcal{A}})\bar{\Gamma}(z^{-1}I - \bar{\mathcal{A}})^T + \bar{\mathcal{A}}\bar{\Gamma}(z^{-1}I - \bar{\mathcal{A}}^T) + (zI - \bar{\mathcal{A}})\bar{\Gamma}\bar{\mathcal{A}}^T = \bar{\mathcal{B}}\bar{\mathcal{B}}^T, \quad (6.15)$$

$$\bar{\Gamma} + (zI - \bar{\mathcal{A}})^{-1}\bar{\mathcal{A}}\bar{\Gamma} + \bar{\Gamma}\bar{\mathcal{A}}^T(z^{-1}I - \bar{\mathcal{A}}^T)^{-1} = (zI - \bar{\mathcal{A}})^{-1}\bar{\mathcal{B}}\bar{\mathcal{B}}^T(z^{-1}I - \bar{\mathcal{A}}^T)^{-1}, \quad (6.16)$$

where $\bar{\Gamma} = \mathbb{E}[\bar{x}_{t-1} \bar{x}_{t-1}^T]$. By pre- and postmultiplying (6.16) by \mathcal{E}_1^T and \mathcal{E}_1 one can attain

$$\begin{aligned} & \mathcal{E}_1^T \bar{\Gamma} \mathcal{E}_1 + \mathcal{E}_1^T (zI - \bar{\mathcal{A}})^{-1} \bar{\mathcal{A}} \bar{\Gamma} \mathcal{E}_1 + \mathcal{E}_1^T \bar{\Gamma} \bar{\mathcal{A}}^T (z^{-1}I - \bar{\mathcal{A}}^T)^{-1} \mathcal{E}_1 \\ &= \mathcal{E}_1^T (zI - \bar{\mathcal{A}})^{-1} \bar{\mathcal{B}} \bar{\mathcal{B}}^T (z^{-1}I - \bar{\mathcal{A}}^T)^{-1} \mathcal{E}_1. \end{aligned} \quad (6.17)$$

Using the results of Theorem 6.3.2 and Theorem 6.3.4, it is obvious that the pairs $[\bar{\mathcal{A}}, \bar{\mathcal{B}}]$ and $[\bar{\mathcal{A}}, \mathcal{E}_1^T]$ respectively are reachable and observable respectively on a generic subset of the parameter space. Thus, $[\bar{\mathcal{A}}, \bar{\mathcal{B}}, \mathcal{E}_1^T]$ is minimal and $\mathcal{E}_1^T(zI - \bar{\mathcal{A}})^{-1}\bar{\mathcal{B}}$ has McMillan degree equal to the dimension of $\bar{\mathcal{A}}$ i.e. $pn - s$. Since the McMillan degree remains unchanged under transposition and replacement of a variable by a Mobius transformation, $\bar{\mathcal{B}}^T(z^{-1}I - \bar{\mathcal{A}}^T)^{-1}\mathcal{E}_1$ has the same McMillan degree. Furthermore, by the stability assumption of the underlying AR system and considering the genericity of the pair $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$, we can conclude that there is no pole-zero cancelation in the product $\mathcal{E}_1^T(zI - \bar{\mathcal{A}})^{-1}\bar{\mathcal{B}}\bar{\mathcal{B}}^T(z^{-1}I - \bar{\mathcal{A}}^T)^{-1}\mathcal{E}_1$.¹ Thus, the McMillan degree of the

¹Observe that if the pair $[\bar{\mathcal{A}}, \bar{\mathcal{B}}]$ is generic, so is the pair $[\bar{\mathcal{A}} + \bar{\mathcal{B}}F, \bar{\mathcal{B}}]$, for any fixed but arbitrary F of the proper dimension. Moreover, while the poles of $\mathcal{E}_1^T(zI - \bar{\mathcal{A}})^{-1}\bar{\mathcal{B}}$ and $\mathcal{E}_1^T(zI - \bar{\mathcal{A}} - \bar{\mathcal{B}}F)^{-1}\bar{\mathcal{B}}$

product $\mathcal{E}_1^T(zI - \bar{\mathcal{A}})^{-1}\bar{\mathcal{B}}\bar{\mathcal{B}}^T(z^{-1}I - \bar{\mathcal{A}}^T)^{-1}\mathcal{E}_1$ is equal to $2(np - s)$. Note that the non-constant terms on the left hand side of (6.17) have the same McMillan degree and share no common poles. Therefore, $\mathcal{E}_1^T(zI - \bar{\mathcal{A}})^{-1}\bar{\mathcal{A}}\bar{\Gamma}\mathcal{E}_1$ has McMillan degree equal to $np - s$. Due to the fact that $[\bar{\mathcal{A}}, \mathcal{E}_1^T]$ is observable, we can easily conclude that the pair $[\bar{\mathcal{A}}, \bar{\mathcal{A}}\bar{\Gamma}\mathcal{E}_1^T]$ is reachable; moreover, $\bar{\mathcal{A}}$ is nonsingular so, the reachability matrix, see Kailath [1980], associated with the pair $[\bar{\mathcal{A}}, \bar{\mathcal{A}}\bar{\Gamma}\mathcal{E}_1^T]$ has the same rank as the reachability matrix associated with the pair $[\bar{\mathcal{A}}, \bar{\Gamma}\mathcal{E}_1^T]$. Hence, we can readily conclude that the pair $[\bar{\mathcal{A}}, \bar{\Gamma}\mathcal{E}_1^T]$ is also reachable. Now recall the definition of K which is $K = \bar{\Gamma} \begin{bmatrix} I_{n_f} \\ 0 \end{bmatrix}$. Based on the definition of \mathcal{E}_1^T , it becomes obvious that we proved the conclusion of theorem for the case where $n_f = 1$ and it is trivial that the result can be generalized for an arbitrary value of n_f . \square

Hence, so far generic identifiability of $[\bar{A}_1 \bar{A}_2 \dots \bar{A}_p]$ and thus of the system parameters $[A_1, \dots, A_p]$ is proved.

6.3.2.2 Generic Identifiability of the Noise Parameters

In this part, it is presented that the noise parameters Σ are generically identifiable.

Theorem 6.3.6. *The noise parameters Σ are generically identifiable from those population second moments which can be observed in principle.*

The proof is analogous to the proof of Theorem 5.3.6 given in the previous chapter.

Note that for generic identifiability we only needed the subsystem $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ to be reachable, which is equivalent to $\bar{\Gamma}$ being nonsingular, whereas in the previous chapter Γ had to be nonsingular for identifiability. Thus we have extended the results of Chapter 5 to the case where we have linear dependencies in x_t , but we have prescribed zero columns in $[A_1, \dots, A_p]$. Also note that for prescribed column degrees $\bar{\Gamma}$ and \mathcal{Z}^f are of the same (full) rank on a generic set.

6.4 AR Systems with Zero Column Degree in $A(q)$

In the previous section, we only considered AR systems whose columns of their AR polynomial matrix, $A(q)$ had unequal prescribed column degrees and no column degree was prescribed to be zero. In this section, it is permitted that column degrees of $A(q)$ are prescribed to be zero. Here, firstly a subsystem from the AR system (6.1) is defined. Then generic identifiability of this subsystem, which turns out to be an

are generically different, their zeros are the same. Now suppose that there exists a z_0 which is both a zero of $\mathcal{E}_1^T(zI - \bar{\mathcal{A}})^{-1}\bar{\mathcal{B}}$ and the reciprocal of an eigenvalue of $\bar{\mathcal{A}}$. Then for almost all F the zeros of $\mathcal{E}_1^T(zI - \bar{\mathcal{A}} - \bar{\mathcal{B}}F)^{-1}\bar{\mathcal{B}}$ will be distinct to the reciprocal of the eigenvalues of $\bar{\mathcal{A}} + \bar{\mathcal{B}}F$ and there will be no pole-zero cancelation.

AR system, using ideas of the previous section, is discussed. Finally, this section ends by explaining how the remainder of parameters can be retrieved.

Accordingly, define a subprocess y_t^r of y_t which consists of those components of y_t not corresponding to those columns of $A(q)$ with prescribed zero column degree.

Attention is first given to identifying those parameters associated with y_t^r . Then later, we obtain the rest of the parameters. Note that in general a marginalized AR process is not an AR process any more, but in the case of zero column degrees, the components of y_t^r are. This is explained in the following lemma.

Lemma 6.4.1. *Consider the AR process as defined in (6.1) and assume that its AR polynomial matrix $A(q)$ has one or more columns with zero degree. Then the process y_t^r obtained from deleting all components of y_t associated with columns of $A(q)$ with zero degree, is an AR process of the same order as (6.1).*

Proof. The AR process (6.1) is $A(q)y_t = be_t$, where $A(q) = I - A_1q - \dots - A_pq^p$. Then we define a matrix V by deleting from $np \times np$ the identity matrix any column j for which the j th-column of $A(q)$ has zero column degree zero. Then define $V^T A(q)V = \tilde{A}(q)$, $V^T y_t = y_t^r$ and $V^T b = \tilde{b}$. It follows then that

$$\tilde{A}(q)y_t^r = \tilde{b}e_t \quad (6.18)$$

Now let $y_t^{(i)}$ represent a component of y_t which is associated with any column of $A(q)$ with degree greater than zero. Observe that it does not depend on any lagged y_t components associated with those columns of $A(q)$ with degree zero. Thus, the system (6.18) represents an AR process associated with y_t^r alone. \square

To illustrate the approach for dealing with the case where one or more columns of $A(q)$ have column degree zero, the following illustrative example is provided. The main results are stated afterwards.

Example 6.4.2. *Consider the following AR(3) process*

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + A_3 y_{t-3} + be_t,$$

where $y_t \in \mathbb{R}^3$ and A_1, A_2 and A_3 have the following structure:

$$A_1 = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} \times & 0 & 0 \\ \times & 0 & 0 \\ \times & 0 & 0 \end{bmatrix}. \quad (6.19)$$

Based on the above discussion we define

$$x_{t+1}^r = \begin{bmatrix} y_t^r \\ y_{t-1}^r \\ y_{t-2}^r \end{bmatrix} = \begin{bmatrix} y_t^{(1)} \\ y_t^{(2)} \\ y_{t-1}^{(1)} \\ y_{t-1}^{(2)} \\ y_{t-2}^{(1)} \\ y_{t-2}^{(2)} \end{bmatrix}, \quad (6.20)$$

where, $y_t^{(i)}$ denotes the i -th component of y_t . Accordingly, the state space equation is

$$x_{t+1}^r = A_r x_t^r + B_r \epsilon_t, \quad (6.21)$$

where

$$A_r = \begin{bmatrix} \times & \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, B_r = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (6.22)$$

The first two rows of A_r are $[\tilde{A}_1 \ \tilde{A}_2 \ \tilde{A}_3]$. Note that the process y_t^r can be obtained only by considering the first two rows of the equation (6.21).

$$y_t^r = \begin{bmatrix} y_t^{(1)} \\ y_t^{(2)} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \epsilon_t = \tilde{A}_1 y_{t-1}^r + \tilde{A}_2 y_{t-2}^r + \tilde{A}_3 y_{t-3}^r + \bar{b} \epsilon_t, \quad (6.23)$$

where $b = [b_1^T \ b_2^T \ b_3^T]^T$, $\bar{b} = [b_1^T \ b_2^T]^T$, $b = [\bar{b}^T \ \hat{b}^T]^T$. For future reference, we note that of the 15 parameters appearing in the A_i , only 10 appear in A_r . We will first deal with their identification.

Since there still exists a zero column in A_r we can further reduce the state. Thus, we define

$$\bar{x}_{t+1}^r = \begin{bmatrix} y_t^{(1)} \\ y_t^{(2)} \\ y_{t-1}^{(1)} \\ y_{t-1}^{(2)} \\ y_{t-2}^{(1)} \end{bmatrix}, \quad (6.24)$$

and the related state space model will be of the form

$$\bar{x}_{t+1}^r = \bar{A}_r \bar{x}_t^r + \bar{B}_r \epsilon_t, \quad (6.25)$$

where

$$\overline{\mathcal{A}}_r = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \overline{\mathcal{B}}_r = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (6.26)$$

The first two rows of $\overline{\mathcal{A}}_r$ are $[\check{A}_1 \check{A}_2 \check{A}_3]$ and there holds

$$y_t^r = [\check{A}_1 \check{A}_2 \check{A}_3] \bar{x}_t^r + \bar{b} \epsilon_t. \quad (6.27)$$

Consider the generalized version of equation (6.27) for the AR(p) case:

$$y_t^r = [\check{A}_1 \check{A}_2 \dots \check{A}_p] \bar{x}_t^r + \bar{b} \epsilon_t. \quad (6.28)$$

We also need to generalize the state space equation (6.25); thus, with a slight abuse of notation we define the state space model associated with (6.28) as:

$$\bar{x}_{t+1}^r = \overline{\mathcal{A}}_r \bar{x}_t^r + \overline{\mathcal{B}}_r \epsilon_t. \quad (6.29)$$

Then using the same procedure introduced in Subsection 6.3.1, one can obtain the matrix below associated with the system (6.28) (the matrix K^r is defined below)

$$\mathcal{Z}^{rf} = [K^r \overline{\mathcal{A}}_r K^r \dots \overline{\mathcal{A}}_r^{p(n-s_1)-s_2-1} K^r], \quad (6.30)$$

where s_1 is the number of prescribed zero columns in $A(q)$ and s_2 is the number of prescribed zero columns in \mathcal{A}_r . Since not all elements of y_t^r are available at all times, we consider those components of y_t^r that are observed at every time instant i.e. the fast components, denoting the associated vector by y_t^{rf} and then $K^r = \mathbb{E} \left[\bar{x}_{t+1}^r \left[y_t^{rf} \right]^T \right]$.

It is apparent that the parameters in $\overline{\mathcal{A}}_i$ can be determined if the matrix \mathcal{Z}^{rf} has full-row rank. Similar to Subsection 6.3.2 we are interested in generic identifiability. Note that the same definition of generic identifiability is followed, but the parameter space is now associated with (6.28).

The following result can be proved in a similar way as Theorem 6.3.5 and with some slight changes to the argument provided in the Subsection 6.3.2.1.

Proposition 6.4.3. *The matrix \mathcal{Z}^{rf} has full-row rank for a set of generic parameter matrices $\check{A}_i, i = 1, 2, \dots, p$.*

Thus, from the above proposition, it readily follows that the parameter matrices \check{A}_i are generically identifiable from those population second moments which can be observed in principle.

Now similar to the previous section, the generic identifiability of the noise parameters associated with the system (6.28) is examined. Let $\bar{\Sigma}$ be the noise covariance matrix corresponding to \bar{b} . Then using a similar argument as in the proof of Theorem

5.3.6, one can prove that $\bar{\Sigma}$ is generically identifiable from those population second moments which can be observed in principle.

Now, we introduce the following numerical example to illustrate the result of Proposition 6.4.3.

Example 6.4.4. Let us consider the equation (6.27) in Example 6.4.2. Suppose that the $n_f = 2$, $n_s = b_1 = b_2 = 1$. Then for a generic choice of matrices \check{A}_i , $i = 1, 2, 3$, one can verify that the matrix Z^{rf} has rank equal to five i.e. it has full-row rank. However, for a nongeneric situation with

$$\check{A}_1 = \check{A}_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad \check{A}_3 = \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix},$$

the matrix Z^{rf} has rank equal to three. Hence, for this nongeneric situation, the method of modified extended Yule-Walker fails in identifying the parameter matrices from those available population second moments.

At this stage, the task is to obtain those system and noise parameters associated with the suppressed parts of the process y_t due to having columns with zero degree in $A(q)$. In Example 6.4.2, there are five such parameters appearing in the last row of the A_i .

Let y_t^f contain all those components of y_t which we deleted when forming y_t^r and \hat{A}_i is obtained from A_i by deleting all prescribed zero columns and taking only the rows associated with y_t^f , \hat{b} consists of the rows b corresponding to the components of y_t^f . The following proposition states that the parameters \hat{A}_i and the corresponding noise parameters are generically identifiable.

Proposition 6.4.5. Let y_t^f denote those components of the process y_t which are associated with columns of $A(q)$ with zero degree. Then the system and noise parameters associated with y_t^f , called $[\hat{A}_1, \hat{A}_2, \dots, \hat{A}_p]$ and $\hat{b}\hat{b}^T$ respectively, are generically identifiable.

Proof. We are starting from

$$y_t^f = [\hat{A}_1, \hat{A}_2, \dots, \hat{A}_p] \bar{x}_t^r + \hat{b}\epsilon_t, \quad (6.31)$$

where \bar{x}_t^r is the vector of all components of $(y_{t-1}, \dots, y_{t-p})$, which do not correspond to zero columns in $A(q)$. We first obtain the system parameters \hat{A}_i ; one can easily verify that:

$$\mathbb{E} [y_t^f \chi_t^f] = [\hat{A}_1, \hat{A}_2, \dots, \hat{A}_p] \underbrace{\mathbb{E} [\bar{x}_t^r \chi_t^f]}_{\bar{\Omega}} + \hat{b} \underbrace{\mathbb{E} [\epsilon_t \chi_t^f]}_0, \quad (6.32)$$

where $\chi_t^f = \begin{bmatrix} y_{t-1}^{fT} & \dots & y_{t-np+s_1}^{fT} \end{bmatrix}$. Now by using the result of Proposition 6.4.3, one can readily conclude that the matrix $\bar{\Omega}$ has full-row rank. Hence, the parameters \hat{A}_i are identifiable. Since all the parameters of the \mathcal{A} are available, it follows from Theorem 2 in Anderson et al. [2012] that the noise parameters Σ are generically identifiable; thus, missing elements of Σ , viz. $\hat{b}\hat{b}^T$ are also generically identifiable. \square

6.5 Summary

In this chapter, a special class of singular autoregressive models were examined. It was demonstrated that vector autoregressions with prescribed column degrees are generically identifiable on a restricted parameter space from covariance data in which significant information is missing, corresponding to the fact that some system outputs are not available every time instant. To obtain this, the modified extended Yule-Walker technique was exploited.

Conclusions and Future Research

In conclusion, the current chapter summarizes the main contributions of the preceding chapters and also provides suggestions for possible research directions. The principal motivation behind this thesis was to use tools from systems and control to solve several important problems associated with linear systems with alternately missing measurements. As mentioned earlier, such systems arise in different fields of science and technology and are of significant importance. The main contributions of each individual chapters are summarized in Section 7.1 and suggestions for possible future research are outlined in section 7.2.

7.1 Contributions of Thesis

- **Chapter 2**

Zeros of the blocked system obtained from the blocking of linear discrete time-invariant systems were studied in this chapter. In particular, for a very first time (without imposing any condition on the structure or normal rank of the blocked system), it was shown that the blocked system is zero-free if and only if the related unblocked system is zero-free. In addition, the system matrix of the blocked system was investigated under the genericity assumption. It was demonstrated that the blocked system generically has no zeros when it is either fat or tall. However, when the blocked system is square, it generically has a finite zero and the kernel associated with that zero is of dimension one.

- **Chapter 3**

Zeros of tall discrete-time multirate linear systems were addressed in this chapter. With the zeros of multirate linear systems being defined as those of their corresponding blocked systems, the system matrix of tall blocked systems was investigated for generic choice of parameter matrices. It was specifically shown that tall blocked systems generically have no finite nonzero zeros. However, we showed that there are situations under which these systems present zeros at $Z = 0$ or $Z = \infty$ or both. Such situations can be characterized in terms of the relevant integer parameters (input, state, and output dimensions) and ratio of sampling rates. As part of the investigation, we also accurately calculated the normal rank assumed by the system matrix of the blocked system.

- **Chapter 4**

Linear multirate systems with coprime measured output rates were studied in this chapter. For their blocked linear time-invariant version, we studied the generic observability and reachability using a two-step blocking approach. Moreover, under the assumption that the parameter matrices are chosen generically, we explored finite nonzero zeros. In particular, the focus was on tall blocked systems. It was demonstrated that there exist three possible regions and combinations of fast and slow output dimensions and sampling rates which lead to a tall blocked system. Here, we only studied two regions and demonstrated sufficient conditions for the blocked system matrix to have full-column normal rank. Then under certain conditions, it was proved that blocked systems generically have no finite nonzero zeros.

- **Chapter 5**

This chapter demonstrated that AR models are generically identifiable from covariance data in which significant information is missing, corresponding to the fact that some system outputs are only available every N -th time instants for some $N > 1$. To obtain this result, the extended Yule-Walker equations proposed by Chen and Zadrozny [1998] were modified. Then the adjusted equations called modified extended Yule-Walker equations were used to prove the generic identifiability result.

- **Chapter 6**

In this chapter, we built on our work in Chapter 5 and demonstrated that vector autoregressions with prescribed column degrees are generically identifiable on a restricted parameter space from covariance data with missing components.

We considered two cases. In case one, we assumed to be prescribed nonzero unequal column degrees and we showed generic identifiability of the system and noise parameters. Then in the second case, we showed generic identifiability for the case of prescribed zero column degrees by dividing the problem into two steps. In step one, we only treated a subsystem corresponding to nonzero prescribed column degrees. It turned out that this subsystem can be treated under the framework of case one and therefore we had generic identifiability for parameter matrices of this subsystem. Then in the second step, we obtained those parameter matrices not included in step one.

7.2 Plan for Future Research

In this section we discuss some possible extensions and further research directions. It is worthwhile mentioning that in the primary application of this thesis i.e. econometric modeling, models are initially set in a discrete-time domain. However, in some other applications such as systems and control, it is also common that the discrete-time model is built from discretization of an underlying continuous-time model. It is desirable that parameters of the discrete-time model capture some specifications of

the associated continuous-time system see e.g. Farina et al. [2013]. In this scenario, the parameters of the obtained discrete-time system may not necessarily be generic. However, several results of this thesis use the genericity assumption. Hence, it is interesting to explore these results when the genericity assumption for parameter matrices does not hold anymore.

- Chapter 2

1. In this chapter, we mainly studied a relationship between zeros of the unblocked system and those of the corresponding blocked system. The multiplicity of zeros may change under the blocking procedure and it requires a more complete study to show how changes happen. This study should cover all types of zeros i.e. finite nonzero zeros, zeros at the origin and zeros at infinity.
2. This chapter also studied blocked systems resulting from unblocked systems with generic parameter matrices. However, in practice one often faces systems following some physical laws, and defining matrices of such systems contain some fixed parameters and some free ones. The notion of *structured systems* are used in the systems and control literature to refer to these systems Dion et al. [2003]. Another problem that can be considered for future research involves the analysis of zeros for blocked systems resulting from the blocking of linear structured systems.
3. The blocking technique studied in this chapter can be helpful for identifying parameters of periodic autoregressive moving-average (PARMA) processes from data. This technique enables us to employ standard identification methods for these processes. It would be interesting to exploit this technique and compare the results with those of some other existing algorithms such as Anderson et al. [2013b] and Adams and Goodwin [1995].

- Chapter 3

1. In this chapter, the blocking approach was used to transfer multirate linear systems into linear time-invariant blocked systems. In the literature, there exists another reformulation termed *cyclic reformulation* for obtaining a linear time-invariant version of a multirate system Bittanti and Colaneri [2009]. It is interesting to study zeros of multirate linear systems with generic parameter matrices using cyclic reformulation. It is important to note that the cyclic reformulation has an advantage over the blocking reformulation as its parameter matrices are less structured.
2. The chapter was more concerned with multirate systems. However, as a theoretical research question, one can investigate the possibility of extending the results of this chapter to periodic systems. This means dealing with systems which are not only periodic in the output and feedthrough matrices but also in the state and input matrices.

3. Another possible research question is to explore multirate linear structured systems. This direction of research may include analysis of zeros and the structural properties of their corresponding blocked or cyclic systems.
4. This chapter studied multirate systems with outputs available at two rates. It would be interesting to generalize the results of this chapter to the case where there exist more than two rates. The author believes that some of the techniques used in this chapter will be helpful for this generalization.

• Chapter 4

1. The results presented in this chapter only captures finite nonzero zeros of a particular class of multirate systems namely those multirate systems in regions one and two of Fig. 4.2 with their associated system matrix having full-column normal rank. However, our simulation results show that the main result of this chapter i.e. Theorem 4.4.10, holds for all regions. Hence, an important open problem here is concerned with the generalization of this result.
2. It is natural to study zeros at the origin and infinity for this case. To this end, several subproblems need to be addressed. These include but are not confined to precisely calculating the normal rank of the corresponding system matrix and obtaining a result which relates the number of zeros at the origin and that of zeros at infinity.

Since the topics of Chapters 5 and 6 are closely related, we discuss open problems related to these two chapters together.

• Chapters 5 and 6

1. These two chapters mainly dealt with generic identifiability of AR systems. It is also essential to address the question of generic identifiability for ARMA models as well.
2. In these two chapters, we used the modified extended Yule-Walker approach to obtain the system parameters from those available second moments. However, there is a need for a comparative examination between the modified extended Yule-Walker estimator and other proposed ones like Mariano and Murasawa [2010] and Ghysels et al. [2004].
3. Another important open question concerns the loss of information due to mixed frequency data, in particular the question of the relative efficiency of the estimator derived for the mixed frequency data from the modified extended Yule-Walker equations in relation to the Yule-Walker estimators for a high frequency system.
4. The theoretical results presented in these chapters should be applied to real world data sets. Such data sets are available from sources like the

European Central Bank (ECB). In close collaboration with our colleagues from the Technical University of Vienna, we have achieved some progress toward this direction. However, there are more steps to be made along this path and the results will be recorded in our future manuscripts.

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